ON THE EXTENSION OF DELAUNAY'S METHOD IN THE LUNAR THEORY TO THE GENERAL PROBLEM OF PLANETARY MOTION*

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PART I.—EXPOSITION OF THE THEORY.

The method of integrating the differential equations of motion, adopted by DELAUNAY for the elaboration of his lunar theory, as it is explained by him, demands its division into several cases, and is established through very tedious transformations. These disadvantages disappear when the greatest generality is given to the procedure. Hence, an explanation of the method, as it would be applied to the motion of a planetary system like the solar, will, doubtless, be welcome to astronomers.

I.

Let T denote the living force of the system, Ω the potential function, and, with POINCARÉ, put

$$F = \Omega - T.$$

The k variables, necessary for completely defining the position of the system, may be denoted by $q_1, q_2, \cdots, q_k$. Use accents to denote complete differentiation with respect to the time of the latter, and we have

$$T = \text{function}\,(q_1, q_2, \cdots, q_k, q_1', q_2', \cdots, q_k').$$

The partial derivatives of this function with respect to the k variables q_i' are to be used as variables instead of the latter, and we put

$$p_i = \frac{\partial T}{\partial q_i'}, \qquad (i = 1, 2, \cdots, k).$$

By means of these k equations the q_i' can be eliminated from T, and thus will result:

$$T = \text{function}\,(q_1, q_2, \cdots, q_k, p_1, p_2, \cdots, p_k).$$

* Presented to the Society February 24, 1900. Received for publication February 5, 1900.

Then the system of differential equations for determining the variables p_i and q_i is:

$$\frac{dp_i}{dt}=\frac{\partial F}{\partial q_i},\quad \frac{dq_i}{dt}=-\frac{\partial F}{\partial p_i},\qquad (i=1,\ 2,\ \cdots,\ k). \tag{1}$$

Let us suppose that Ω is separated into the two parts Ω_0 and Ω_1, and that, when we neglect Ω_1 in F, the equations (1) can be completely integrated, and their integrals expressed in terms of two sets of k quantities each, symbolized thus:

$$L_1,\ L_2,\ \cdots,\ L_k,$$
$$\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k,$$

of which the first set are constants, and the second set linear functions of the time of the form $n_i t + c_i$, n_i being a function of the L_i, and c_i an arbitrary constant. Nothing forbids our taking the L_i such that they may be the elements severally conjugate to the λ_i.

Now, desiring to integrate the equations (1) when F has its complete value, we may adopt the L_i and the λ_i as the dependent variables to be employed. The differential equations of the problem are then:

$$\frac{dL_i}{dt}=\frac{\partial F}{\partial \lambda_i},\quad \frac{d\lambda_i}{dt}=-\frac{\partial F}{\partial L_i},\qquad (i=1,\ 2,\ \cdots,\ k). \tag{2}$$

Here the function F has been made to involve the L_i and λ_i by eliminating the old variables p_i and q_i from it by means of their values given by the integrals derived on the supposition that $\Omega=\Omega_0$. As

$$F= \text{a constant}$$

is an integral of the problem, and $\Omega_0 - T=$ a constant, when Ω_1 is neglected, it is quite evident that when we substitute in $\Omega_0 - T$ for p_i and q_i their values in terms of the L_i and λ_i, the λ_i completely disappear and $\Omega_0 - T$ becomes a function of the L_i only. Thus, in the second form for F, the variables λ_i enter into it solely through the portion Ω_1.

II.

In order to exemplify we will adduce the solar system composed of the Sun and the eight major planets. We will suppose that the masses of the Sun, Mercury, Venus, $\cdots$, Neptune are denoted by m_0, m_1, m_2, $\cdots$, m_8, and will put

$$\mu_i = m_0 + m_1 + m_2 + \cdots + m_i,\quad \kappa_i=\frac{m_i}{\mu_i},\qquad (i=0,\ 1,\ \cdots,\ 8).$$

Let the type of representation of the rectangular coördinates of the planets relative to the Sun be as follows:

Mercury	x_1,
Venus	$x_2 + \kappa_1 x_1$,
Earth	$x_3 + \kappa_2 x_2 + \kappa_1 x_1$,
. .	. . . .
Neptune	$x_8 + \kappa_7 x_7 + \cdots + \kappa_1 x_1$.

The differential equations these variables satisfy are:

$$(3)\qquad \begin{cases} \mu_{i-1}\kappa_i \dfrac{d^2x_i}{dt^2} = \dfrac{\partial\Omega}{\partial x_i}, \\ \mu_{i-1}\kappa_i \dfrac{d^2y_i}{dt^2} = \dfrac{\partial\Omega}{\partial y_i}, \\ \mu_{i-1}\kappa_i \dfrac{d^2z_i}{dt^2} = \dfrac{\partial\Omega}{\partial z_i}, \end{cases} \qquad (i=1,\ 2,\ \cdots,\ 8).$$

Here Ω denotes the sum of the products of every two masses of the system divided by their distance, a relation we will write thus:

$$(4)\qquad \Omega = m_0 \sum \frac{m_i}{\Delta_{0,i}} + \sum \frac{m_i m_j}{\Delta_{i,j}}.$$

Suppose that the portion to be separated from Ω is

$$(5)\qquad \Omega_0 = m_0 \sum \frac{m_i}{r_i},$$

r_i standing for $\sqrt{(x_i^2 + y_i^2 + z_i^2)}$. Then, if Ω_0 is substituted for Ω in equations (3), and the members are divided by $\mu_{i-1}\kappa_i$, we get

$$(6)\qquad \begin{cases} \dfrac{d^2x_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{x_i}{r_i^3} = 0, \\ \dfrac{d^2y_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{y_i}{r_i^3} = 0, \\ \dfrac{d^2z_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{z_i}{r_i^3} = 0, \end{cases} \qquad (i=1,\ 2,\ \cdots,\ 8).$$

It will be seen that each group of these equations, corresponding to the same value of i, is independent of all the rest, and that it differs from the group of equations of relative motion of two bodies only in that the constant $m_0\ \mu_i/\mu_{i-1}$ takes the place of $m_0 + m_i$.

Let a_i be the semi-axis major, e_i the eccentricity, ϕ_i the inclination, l_i the mean anomaly, g_i the angular distance of the perihelion from the node, and h_i the longitude of the node. Put

$$
(7) \qquad \begin{cases} L_i = \sqrt{m_0 \dfrac{\mu_i}{\mu_{i-1}} a_i}\,, \\ G_i = L_i \sqrt{1 - e_i^2}\,, \\ H_i = G_i \cos \phi_i\,, \end{cases} \qquad (i = 1, 2, \cdots, 8).
$$

Then, when the elements become variable by reason of the addition of Ω_1 to Ω_0, they will satisfy the differential equations:

$$
(8) \qquad \begin{cases} \dfrac{dL_i}{dt} = \dfrac{\partial R_i}{\partial l_i}, & \dfrac{dl_i}{dt} = -\dfrac{\partial R_i}{\partial L_i}, \\ \dfrac{dG_i}{dt} = \dfrac{\partial R_i}{\partial g_i}, & \dfrac{dg_i}{dt} = -\dfrac{\partial R_i}{\partial G_i}, \\ \dfrac{dH_i}{dt} = \dfrac{\partial R_i}{\partial h_i}, & \dfrac{dh_i}{dt} = -\dfrac{\partial R_i}{\partial H_i}, \end{cases} \qquad (i = 1, 2, \cdots, 8),
$$

where R_i will be, in terms of Ω_1, mentioned above,

$$
(9) \qquad R_i = \frac{m_0 \dfrac{\mu_i}{\mu_{i-1}}}{2a_i} + \frac{\Omega_1}{\mu_{i-1}\kappa_i}, \qquad (i = 1, 2, \cdots, 8).
$$

Desiring to have the same perturbative function, whatever may be the integer i, we multiply the values (7) of L_i, G_i, H_i, as also the value (9) of R_i by the constant $\mu_{i-1}\kappa_i$, which does not alter the form of equations (8). We now have:

$$
(10) \qquad \begin{cases} L_i = m_i \sqrt{m_0 \dfrac{\mu_{i-1}}{\mu_i} a_i}\,, \\ G_i = L_i \sqrt{1 - e_i^2}\,, \\ H_i = G_i \cos \phi_i\,, \end{cases} \qquad (i = 1, 2, \cdots, 8),
$$

as also:

$$
(11) \qquad F = m_0 \sum_{i=1}^{i=8} \frac{m_i}{2a_i} + m_0 \sum_{i=2}^{i=\infty} m_i \left[\frac{1}{\Delta_{0,i}} - \frac{1}{r_i} \right] + \sum \frac{m_i m_j}{\Delta_{i,j}}.
$$

If the planetary coördinates in the last equation are replaced by the elements (10) and the l_i, g_i, h_i, the differential equations of the system are:

$$(12)\qquad \begin{cases} \dfrac{dL_i}{dt} = \dfrac{\partial F}{\partial l_i}; & \dfrac{dl_i}{dt} = -\dfrac{\partial F}{\partial L_i}, \\ \dfrac{dG_i}{dt} = \dfrac{\partial F}{\partial g_i}, & \dfrac{dg_i}{dt} = -\dfrac{\partial F}{\partial G_i}, \\ \dfrac{dH_i}{dt} = \dfrac{\partial F}{\partial h_i}, & \dfrac{dh_i}{dt} = -\dfrac{\partial F}{\partial H_i}, \end{cases} \qquad (i = 1, 2, \cdots, 8).$$

In the second term of the right member of (11) the quantities $1/\Delta_{0,2}$, $1/\Delta_{0,3}$, $\cdots$, $1/\Delta_{0,8}$, can be developed in infinite series, the first terms of which are $1/r_2$, $1/r_3$, $\cdots$, $1/r_8$, and thus are cancelled by the term $1/r_i$. Then the two latter terms of (11) are of the second order with reference to planetary masses.

III.

In order to make the application of DELAUNAY'S method it appears necessary that F should be developed in a series, finite or infinite and periodic with respect to the variables l_i, g_i, h_i, which have been named the angular variables. In astronomical problems the series are generally infinite. For legitimate employment this series must remain convergent throughout the whole duration of motion, while t is passing from $-\infty$ to $+\infty$. It becomes then pertinent to ask what conditions must be fulfilled in order that this series may be convergent. It is well known that the reciprocal of the distance between two planets can be developed in a convergent infinite series, periodic with respect to the mean anomalies of the planets, provided that the orbits, as they stand in space, have no point in common, or when the reciprocal of the distance never becomes infinite. The condition of convergence in the present case is precisely similar to this. Here, however, not only the mean anomalies l_i are left indeterminate in the series, but also the remaining angular variables g_i and h_i which define the positions of the perihelia and nodes. Hence, in the present case, there must not only be no actual intersection of the orbits, but none when the perihelia and nodes are shifted in every possible way, the linear variables, or the mean distances, eccentricities and inclinations retaining their actual values. In the Delaunay development of the reciprocal of the distance between two planets, it is necessary and it suffices for convergence that the perihelion radius of one of the planets should always exceed the aphelion radius of the other.

We may consider this subject under a more general aspect. Let F have the periodic development

$$(13)\qquad F = \sum A \cos\,[j_1\lambda_1 + j_2\lambda_2 + \cdots + j_k\lambda_k],$$

where $\lambda_1 \cdots \lambda_k$ are the angular variables, the j positive or negative integers, and A is a function of the linear variables L only. That this infinite series may be convergent, F must not only actually never become infinite, but never even potentially so. It is necessary here to explain what we mean by the qualifying epithet "potentially." If, while the linear variables L are supposed to maintain their actual values, and, consequently, the coefficients A their actual values, we allow to all the angular variables λ the complete swing of movement from 0 to 2π, F remains always finite, we say it never *potentially* becomes infinite. In order that F may not *actually* become infinite it is necessary and sufficient that the velocities of and the distances between the points of the system should remain finite. In order that F may not *potentially* become infinite, it is necessary and sufficient that the values of the linear variables $L_1, L_2, \cdots, L_k$ should remain within a certain domain. The definition of this domain is very complex after a Delaunay transformation has been operated, but is quite simple in terms of the original Keplerian linear variables $L_1, L_2, \cdots, L_k$.

We may illustrate this subject by bringing forward the case of the solar system as it has been described in § II. Employing the linear elements a_i and e_i of equation (10) or of equations (7), the inequalities which define this domain are:

$$(14)\qquad \begin{cases} a_1(1-e_1) > 0, \\ a_2(1-e_2) - (1-\kappa_1)\,a_1(1+e_1) > 0, \\ a_3(1-e_3) - (1-\kappa_2)\,a_2(1+e_2) > 0, \\ \cdots\cdots\cdots\cdots \\ a_8(1-e_8) - (1-\kappa_7)\,a_7(1+e_7) > 0, \\ a_8(1+e_8) < \infty, \end{cases}$$

or, if we prefer to use the elements L_i and G_i of ~~the same equations~~ (10), the inequalities are:

$$(15)\qquad \begin{cases} L_1[L_1 - \sqrt{L_1^2 - G_1^2}] > 0, \\ \dfrac{\mu_2}{m_2^2} L_2[L_2 - \sqrt{L_2^2 - G_2^2}] > \dfrac{\mu_1}{m_1^2} L_1[L_1 + \sqrt{L_1^2 - G_1^2}], \\ \dfrac{\mu_3}{m_3^2} L_3[L_3 - \sqrt{L_3^2 - G_3^2}] > \dfrac{\mu_2}{m_2^2} L_2[L_2 + \sqrt{L_2^2 - G_2^2}], \\ \cdots\cdots\cdots\cdots \\ \dfrac{\mu_8}{m_8^2} L_8[L_8 - \sqrt{L_8^2 - G_8^2}] > \dfrac{\mu_7}{m_7^2} L_7[L_7 + \sqrt{L_7^2 - G_7^2}], \\ L_8 + \sqrt{L_8^2 - G_8^2} < \infty. \end{cases}$$

It is remarkable that the inclinations of the orbits play no part in these inequalities.

IV.

As, in general, it is not necessary to distinguish between the three kinds l_i, g_i, h_i of angular variables, nor between the three kinds L_i, G_i, H_i, of linear variables, for simplicity of notation we shall suppose that the angular variables are denoted by $l_1, l_2, \cdots, l_k$, and their corresponding conjugate linear variables by $L_1, L_2, \cdots, L_k$.

Selecting a particular linear combination θ of of the angular variables so that

$$\theta = j_1 l_1 + j_2 l_2 + \cdots + j_k l_k,$$

the j being positive or negative integers prime to each other, Delaunay's method, somewhat generalized, consists in making such a transformation of variables as would constitute a complete solution of the problem if F, in its periodic development, contained as arguments only integral multiples of θ. That is, in this special case, the new linear variables would turn out constants, and the new angular variables would be of the form $n(t+c)$, n and c being likewise constants. It is clear that when we make such a transformation in F, the terms in the former periodic development involving the cosines of the finite multiples of θ will disappear, but the absolute term will receive a modification. A little consideration will make it evident that the derivation of such a transformation is dependent on quadratures.

The discussion of this derivation is greatly facilitated by making a linear transformation of variables, in the cases of both angular and linear variables. In this it is evident that we can take θ as one of the angular variables; then let Θ be its conjugate linear variable. Thus, we may have the following as the variables involved in the problem:

$$\text{Linear variables,}\quad \Theta,\ \Lambda_1,\ \Lambda_2,\ \cdots,\ \Lambda_{k-1};$$
$$\text{Angular variables,}\quad \theta,\ \lambda_1,\ \lambda_2,\ \cdots,\ \lambda_{k-1}.$$

And the canonical system of differential equations will be:

$$(16)\qquad \begin{cases} \dfrac{d\Theta}{dt} = \dfrac{\partial F}{\partial \theta}, & \dfrac{d\theta}{dt} = -\dfrac{\partial F}{\partial \Theta}, \\[2ex] \dfrac{d\Lambda_1}{dt} = \dfrac{\partial F}{\partial \lambda_1}, & \dfrac{d\lambda_1}{dt} = -\dfrac{\partial F}{\partial \Lambda_1}, \\[2ex] \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\[1ex] \dfrac{d\Lambda_{k-1}}{dt} = \dfrac{\partial F}{\partial \lambda_{k-1}}, & \dfrac{d\lambda_{k-1}}{dt} = -\dfrac{\partial F}{\partial \Lambda_{k-1}}. \end{cases}$$

Let us now consider the mean value of the function F relatively to the angular variables $\lambda_1, \lambda_2, \cdots \lambda_{k-1}$. Then, since F as a periodic function involves only cosines of arguments, if $[F]$ denote the mentioned mean value, we shall have

$$(17) \qquad [F] = \left[\frac{1}{\pi}\int_0^{\pi}\right]^{k-1} F d\lambda_1 d\lambda_2 \cdots d\lambda_{k-1},$$

where the first factor of the right member denotes an operation repeated $k-1$ times, once in reference to each of the variables $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$. As F remains finite whatever values $\theta, \lambda_1, \lambda_2, \cdots, \lambda_k$ may assume, it follows that $[F]$ is finite whatever may be the value of θ. Thus $[F]$ is developable as a periodic function of θ involving only cosines; and we may write

$$(18) \qquad [F] = -B - A_1 \cos\theta - A_2 \cos 2\theta - A_3 \cos 3\theta - \cdots,$$

where $B, A_1, A_2, A_3, \cdots$ are functions of the linear variables $\Theta, \Lambda_1, \Lambda_2, \cdots, \Lambda_{k-1}$.

Let us now suppose that, in equations (16), $[F]$ is substituted for F. They then become:

$$(19) \qquad \left\{\begin{array}{llll} \dfrac{d\Theta}{dt} = \dfrac{\partial[F]}{\partial\theta}, & & \dfrac{d\theta}{dt} = -\dfrac{\partial[F]}{\partial\Theta}, \\ \dfrac{d\Lambda_1}{dt} = 0, & & \dfrac{d\lambda_1}{dt} = -\dfrac{\partial[F]}{\partial\Lambda_1}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{d\Lambda_{k-1}}{dt} = 0, & & \dfrac{d\lambda_{k-1}}{dt} = -\dfrac{\partial[F]}{\partial\Lambda_{k-1}}. \end{array}\right.$$

$\Lambda_1, \Lambda_2, \cdots, \Lambda_{k-1}$ are therefore constants, and the two equations of the first line contain no other variables than Θ and θ, and thus form a distinct system by themselves and determine these two variables; after which, by substitution of values, the remaining differential equations for $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$ determine these variables through quadratures.

V.

As $[F]$ involves only two variables Θ and θ, the two equations which begin (19) have the integral, C being an arbitrary constant,

$$(20) \qquad [F] + C = 0.$$

This integral constitutes a relation between the two variables Θ and θ; and, if the latter are regarded as coördinates defining the position of a point in a plane, (20) is the equation of a plane curve. For this graphical exhibition of the connection between the two variables, we might adopt that in which they are the polar coördinates of a point, Θ being the radius and θ the angle. But, in some cases Θ may pass through zero. This difficulty may be obviated by adding to it a sufficiently large positive constant and thus it be rendered uniformly positive. This can be done provided it does not go to negative infinity. However, all circumstances considered, it will probably be a better course to adopt a representation in rectangular coördinates, θ being the abscissa and Θ the ordinate.

If we derive from (20) an expression for θ in terms of Θ and substitute it in the first equation of (19) and take the reciprocal of both members we shall have the time in terms of Θ by a quadrature, and, by the inversion of this, Θ as a function of t. On the other hand, if we derive from (20) an expression for Θ in terms of θ, and substitute it in the second equation of (19) we shall have the time in terms of θ by a quadrature, and, by the inversion of this, θ as a function of t.

We proceed to note some of the properties of the curve whose equation is (20). In the first place it must be stated that if the differential equations of (19), which determine the variables Θ and θ, compel the first of these to take on values rendering the right member of (18) a divergent series, we agree to set aside such cases as nugatory. Singularities of a certain kind are therefore excluded. The curve cannot have a *point d'arrêt*, for, at this point, we should have simultaneously $d\Theta/dt = 0$ and $d\theta/dt = 0$; and in consequence all succeeding derivatives of these variables would vanish. Thus, at this point Θ and θ would be invariable, which is impossible. It cannot have a multiple point, since, for given values of Θ and θ, there is but one value of each of the quantities $d\Theta/dt$ and $d\theta/dt$. If the curve pass through a point, it must proceed thence until it returns to that point or goes on to infinity. In the latter case, taking a polar representation for the moment, it may either have two infinite branches, or may make an infinite number of turns about the pole, or, in other words be a spiral. But, since equation (20) involves only cosines of θ without sines of the same, the curve must needs be symmetrically situated with respect to the axis from which θ is measured. Hence, the last supposition must be rejected; that is, it cannot be a spiral, nor can it have more than one distinct turn about the pole.

The curves graphically representing (20) may be divided into three classes. Here, for convenience, we adopt a rectangular representation. Let us suppose that an infinite number of values between the limits 0 and π are substituted for θ in (20); the result will be an infinite number of equations for determining the corresponding values of Θ. Let one of these be satisfied by a real value of Θ. Then it may happen that all the remaining equations are satisfied by real values of this variable continuous among themselves and with the value first mentioned. The variable θ can then move from $-\infty$ to $+\infty$ and there will always be a corresponding real value for Θ. The first equation of (19) shows that Θ will be at a maximum or minimum when $\theta = i\pi$, i being a positive or negative integer. As, in the equation

$$(21) \qquad \frac{d\Theta}{dt} = A_1 \sin\theta + 2A_2 \sin 2\theta + 3A_3 \sin 3\theta + \cdots,$$

the quantity A_1 is, in general, larger than A_2, A_3, $\cdots$, it follows that Θ will have no other maximum or minimum values than those just mentioned. In addition, if a maximum value occurs for $\theta = 2i\pi$, then will a minimum value occur for $\theta = (2i + 1)\pi$, and *vice versa.* If, in (20) we put, in succession $\theta = 0$, $\theta = \pi$, we shall have the two equations:

$$(22) \qquad \begin{cases} C - B = \quad A_1 + A_2 + A_3 + \cdots, \\ C - B = -A_1 + A_2 - A_3 + \cdots. \end{cases}$$

And if Θ be regarded as the unknown to be determined by them, it is plain that the maximum value of Θ will be a root of one of them and the minimum value a root of the other. Again Θ cannot be constant unless all the coefficients A vanish. It is quite evident that, in this case, the values of Θ and θ can be represented by the infinite periodic series:

$$(23) \qquad \begin{cases} \Theta = \quad \Theta_0 \quad + \Theta_1 \cos[\theta_0(t + c)] + \Theta_2 \cos 2[\theta_0(t + c)] + \cdots, \\ \theta = \theta_0(t + c) + \theta_1 \sin[\theta_0(t + c)] + \theta_2 \sin 2[\theta_0(t + c)] + \cdots. \end{cases}$$

These two equations are to be regarded as the integrals of the first and second differential equations of the group (19); c is one of the arbitrary constants introduced by the integration, the other may be supposed to be either the C of (20) or the Θ_0 of the first of (23). But while C and c are conjugate to each other, this is not necessarily the case with the elements Θ_0 and $\theta_0(t + c)$. The remaining coefficients of (23), Θ_1, Θ_2, $\cdots$, θ_0, θ_1, θ_2, $\cdots$, are functions of C or Θ_0. On account of the form of the curve which represents (20) in this case it may be called the sinusoid case.

We come now to consider the second case of the representation of (20) by a curve. Here, if we give to θ its range of values between 0 and π, we shall find that the equations determining the corresponding values of Θ have two real roots for an arc of values for θ which either begins at 0 or ends at π; and, in the first case the arc terminates, or, in the second case, begins, at the same intermediate point. At this point the two real roots become equal, and, for the remainder of the semi-circumference, they are imaginary. Consequently, at this point, θ attains either a maximum or minimum value. Because the equation contains only cosines of multiples of θ, in the one case, the right line $\theta = 0$, and, in the other, the right line $\theta = \pi$, divides the area embraced by the curve symmetrically. The maximum and minimum values of Θ are given by the roots of that one of the two equations of (22) which has two real roots. In this case, θ can not be represented by series like the second of (23), but, in general, we may give the integrals of the problem the form :

$$(24)\qquad \begin{cases} \Theta \cos \theta = P_0 + P_1 \cos [\theta_0(t+c)] + P_2 \cos 2[\theta_0(t+c)] + \cdots, \\ \Theta \sin \theta = \qquad Q_1 \sin [\theta_0(t+c)] + Q_2 \sin 2[\theta_0(t+c)] + \cdots, \end{cases}$$

where θ_0, P_0, P_1, P_2, $\cdots$, Q_1, Q_2, $\cdots$, are constant coefficients and functions of the C of (20), while c, as before, is the other arbitrary constant. It will be perceived that, in the former case, the integral equations (23) can be given the form (24) if one chooses ; and DELAUNAY has always adopted it where the eccentricity e would appear as a divisor in the first form. At the two points, at which θ has attained its maximum or minimum value, we have $d\theta/dt = 0$, or

$$(25)\qquad \frac{dB}{d\Theta} + \frac{dA_1}{d\Theta} \cos \theta + \frac{dA_2}{d\Theta} \cos 2\theta + \cdots = 0 .$$

When $dB/d\Theta$ and $dA_1/d\Theta$ are quantities of the same order of magnitude, the second case is likely to occur. As the curve, which here represents the connection of the variables Θ and θ, is a closed one, this case may be called the ovaloid case. This kind of motion in the variables is, however, generally termed a libration. Observation has not yet shown that it occurs in the system of the eight major planets of the solar system, although it is possible it may exist for very large values of the integers j_i. However, should this prove true, the influence of this circumstance on the motion of the system would be quite insignificant.

The third case in the graphical representation of (20) occurs when, in a certain range of values for θ, bisected by the value $\theta = 0$ or by the value $\theta = \pi$, we find a real value for Θ, but this value tends towards positive or negative infinity as the limits are approached. Here there is one maximum and no minimum for Θ or one minimum and no maximum. As in the previous cases, these values occur when $\theta = 0$ or $\theta = \pi$. As long as the instantaneous orbits of the planets composing the system are elliptic in their nature this case cannot present itself. And Θ cannot go beyond a certain limit without some of the elements becoming imaginary. In order, therefore, to prevent the occurrence of functions of complex variables, a modified system has to be adopted. But an illustration of this case can very easily be constructed. In order to escape the difficulty of divergence when $|\Theta|$ exceeds a certain limit, let us suppose that $[F]$ is finite and does not run into an infinite series, and that all the quantities A_i beyond A_1 vanish. Then the equation (20), being solved with reference to $\cos \theta$, gives

$$\cos \theta = \frac{C - B}{A_1} = \frac{f(\Theta)}{F(\Theta)} .$$

Let Θ_0 be the value of Θ when $\theta = 0$; in order to have the present case we ought to have

$$\frac{f(\Theta_0)}{F(\Theta_0)} = 1, \qquad \frac{f(\infty)}{F(\infty)} = \alpha,$$

α being less than unity. We may suppose that Θ is involved linearly in B and A_1, so that, a, b, c, d being constants,

$$f(\Theta) = a + b\Theta, \quad F(\Theta) = c + d\Theta.$$

Then

$$\Theta_0 = \frac{a - c}{d - b}, \qquad \alpha = \frac{b}{d}.$$

All the conditions will be fulfilled if we put

$$f(\Theta) = 3 + 4\Theta, \qquad F(\Theta) = 2 + 5\Theta;$$

whence $\Theta_0 = 1$ and $\alpha = \frac{4}{5}$. Θ is thus continuous while θ is contained between the two values given by the equation $\cos\theta = \frac{4}{5}$. At the limits Θ becomes infinite. In a system of polar coördinates, if Θ is the radius and θ the angle, the equation of the curve graphically exhibiting the connection of the variables Θ and θ is:

$$\cos\theta = \frac{3 + 4\Theta}{2 + 5\Theta}, \text{ or } \Theta = \frac{-3 + 2\cos\theta}{4 - 5\cos\theta}.$$

It is thus a quartic algebraic curve whose equation in rectangular coördinates is:

$$[2x - 4(x^2 + y^2)]^2 = (3 - 5x)^2(x^2 + y^2),$$

whose course resembles that of a hyperbola. The formula for the time is:

$$t + c = \int \frac{d\Theta}{\sqrt{(2 + 5\Theta)^2 - (3 + 4\Theta)^2}} = \frac{1}{3}\int \frac{d\Theta}{\sqrt{(\Theta - \frac{2}{9})^2 - (\frac{7}{9})^2}}.$$

If this be integrated between the limits $\Theta = 1$ and $\Theta = \Theta$ it will give the time required to describe the curve from the point $\theta = 0$ to the point having the radius Θ.

VI.

Let us now suppose that by the integration of the system of differential equations (19), it is proposed to remove from F the periodic terms having the argument θ, that is, those contained in $[F]$. We confine ourselves to the first case as that will usually be the one which presents itself. The integrals of (19) will evidently have the form:

$$
(26)\quad \begin{cases}
\Theta = \Theta_0 + \Theta_1 \cos[\theta_0(t+c)] + \Theta_2 \cos 2[\theta_0(t+c)] + \cdots, \\
\Lambda_1 = \text{a constant}, \quad \Lambda_2 = \text{a constant}, \quad \cdots, \quad \Lambda_{k-1} = \text{a constant}, \\
\theta = \theta_0(t+c) + \theta_1 \sin[\theta_0(t+c)] + \theta_2 \sin 2[\theta_0(t+c)] + \cdots, \\
\lambda_1 = (\lambda_1) + \lambda_1^{(0)}(t+c) + \lambda_1^{(1)} \sin[\theta_0(t+c)] + \lambda_1^{(2)} \sin 2[\theta_0(t+c)] + \cdots, \\
\lambda_2 = (\lambda_2) + \lambda_2^{(0)}(t+c) + \lambda_2^{(1)} \sin[\theta_0(t+c)] + \lambda_2^{(2)} \sin 2[\theta_0(t+c)] + \cdots, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\lambda_{k-1} = (\lambda_{k-1}) + \lambda_{k-1}^{(0)}(t+c) + \lambda_{k-1}^{(1)} \sin[\theta_0(t+c)] + \lambda_{k-1}^{(2)} \sin 2[\theta_0(t+c)] + \cdots,
\end{cases}
$$

where c, (λ_1), (λ_2), $\cdots$, (λ_{k-1}), Λ_1, Λ_2, $\cdots$, Λ_{k-1}, and one other involved in Θ_0, Θ_1, $\cdots$, θ_0, $\lambda_1^{(0)}$, $\lambda_1^{(1)}$, $\cdots$, complete the number of $2k$ arbitrary constants. If then we take formulas of transformation as follows, replacing

$$
(27)\quad \begin{cases}
\Theta \quad \text{by} \quad \Theta_0 + \Theta_1 \cos\theta + \Theta_2 \cos 2\theta + \cdots, \\
\Lambda_1 \quad \text{by} \quad \Lambda_1, \\
\cdots\cdots \\
\Lambda_{k-1} \quad \text{by} \quad \Lambda_{k-1}, \\
\theta \quad \text{by} \quad \theta + \theta_1 \sin\theta + \theta_2 \sin 2\theta + \cdots, \\
\lambda_1 \quad \text{by} \quad \lambda_1 + \lambda_1^{(1)} \sin\theta + \lambda_1^{(2)} \sin 2\theta + \cdots, \\
\cdots\cdots\cdots\cdots \\
\lambda_{k-1} \quad \text{by} \quad \lambda_{k-1} + \lambda_{k-1}^{(1)} \sin\theta + \lambda_{k-1}^{(2)} \sin 2\theta + \cdots,
\end{cases}
$$

the terms of F involving θ as the argument will disappear from it, their mean values going over into the non-periodic term $-B$.

We shall now have a new set of angular variables θ, λ_1, λ_2, $\cdots$, λ_{k-1}, and it will be pertinent to inquire what are the linear variables conjugate to them, so that we may still have differential equations of the canonical form. Taking the two equations of (27) which define Θ and θ in terms of the new variables, let us suppose that the latter are Θ_0 and θ' (an accent is applied to the latter for the sake of discrimination). Then we have

$$
(28)\quad \begin{cases}
\Theta = \Theta_0 + \Theta_1 \cos\theta' + \Theta_2 \cos 2\theta' + \cdots, \\
\theta = \theta' + \theta_1 \sin\theta' + \theta_2 \sin 2\theta' + \cdots.
\end{cases}
$$

Then, by the substitution of these values in the two differential equations of (16) which determine Θ and θ, we get

$$
(29)\quad \begin{cases}
\dfrac{\partial\Theta}{\partial\Theta_0}\dfrac{d\Theta_0}{dt} + \dfrac{\partial\Theta}{\partial\theta'}\dfrac{d\theta'}{dt} = \dfrac{\partial F}{\partial\Theta_0}\dfrac{\partial\Theta_0}{\partial\theta} + \dfrac{\partial F}{\partial\theta'}\dfrac{\partial\theta'}{\partial\theta}, \\[2ex]
\dfrac{\partial\theta}{\partial\Theta_0}\dfrac{d\Theta_0}{dt} + \dfrac{\partial\theta}{\partial\theta'}\dfrac{d\theta'}{dt} = -\dfrac{\partial F}{\partial\Theta_0}\dfrac{\partial\Theta_0}{\partial\Theta} - \dfrac{\partial F}{\partial\theta'}\dfrac{\partial\theta'}{\partial\Theta}.
\end{cases}
$$

By multiplying these equations by the proper factors, and putting Δ for the functional determinant or Jacobian:

$$\Delta = \frac{\partial\Theta}{\partial\Theta_0}\frac{\partial\theta}{\partial\theta'} - \frac{\partial\Theta}{\partial\theta'}\frac{\partial\theta}{\partial\Theta_0}, \tag{30}$$

we have

$$\Delta\frac{d\Theta_0}{dt} = \frac{\partial F}{\partial\theta'}, \qquad \Delta\frac{d\theta'}{dt} = -\frac{\partial F}{\partial\Theta_0}. \tag{31}$$

But

$$\begin{aligned}\Delta = &\left[1 + \frac{\partial\Theta_1}{\partial\Theta_0}\cos\theta' + \frac{\partial\Theta_2}{\partial\Theta_0}\cos 2\theta' + \cdots\right]\left[1 + \theta_1\cos\theta' + 2\theta_2\cos 2\theta' + \cdots\right] \\ &+ \left[\Theta_1\sin\theta' + 2\Theta_2\sin 2\theta' + \cdots\right]\left[\frac{\partial\theta_1}{\partial\Theta_0}\sin\theta' + \frac{\partial\theta_2}{\partial\Theta_0}\sin 2\theta' + \cdots\right].\end{aligned} \tag{32}$$

According to the theorem of POISSON, Δ is independent of t, or what in this case amounts to the same thing, of θ'. Hence, in computing its value, we have regard only to the absolute term. Thus

$$\begin{aligned}\Delta &= 1 + \tfrac{1}{2}\left[\theta_1\frac{\partial\Theta_1}{\partial\Theta_0} + 2\theta_2\frac{\partial\Theta_2}{\partial\Theta_0} + \cdots + \Theta_1\frac{\partial\theta_1}{\partial\Theta_0} + 2\Theta_2\frac{\partial\theta_2}{\partial\Theta_0} + \cdots\right] \\ &= 1 + \tfrac{1}{2}\frac{\partial}{\partial\Theta_0}\left[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots\right].\end{aligned} \tag{33}$$

Then, if we adopt a new variable Θ' in place of Θ_0 such that

$$\Theta' = \int \Delta d\Theta_0 = \Theta_0 + \tfrac{1}{2}\left[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots\right], \tag{34}$$

equations (31) will be transformed into:

$$\frac{d\Theta'}{dt} = \frac{\partial F}{\partial\theta'}, \qquad \frac{d\theta'}{dt} = -\frac{\partial F}{\partial\Theta'}, \tag{35}$$

which have the canonical form. As to the remaining linear variables Λ_1, Λ_2, $\cdots$, Λ_{k-1}, which are identical with the former variables denoted by the same symbols, it is evident that they remain the conjugates of the new variables λ_1, λ_2, $\cdots$, λ_{k-1}.

VII.

As it is somewhat difficult to discover the linear transformation of variables required to pass from the set

$$\left.\begin{matrix} L_1, & L_2, & \cdots, & L_k \\ l_1, & l_2, & \cdots, & l_k \end{matrix}\right\} \text{ to the set } \left.\begin{matrix} \Theta, & \Lambda_1, & \Lambda_2, & \cdots, & \Lambda_{k-1} \\ \theta, & \lambda_1, & \lambda_2, & \cdots, & \lambda_{k-1} \end{matrix}\right\},$$

suppose we adhere to the use of the first set. Then we have the differential equations

$$\frac{d\Theta}{dt} = \frac{\partial[F]}{\partial\theta}, \quad \frac{d\theta}{dt} = -\frac{\partial[F]}{\partial\Theta}, \tag{36}$$

$$\frac{dL_i}{dt} = j_i\frac{\partial[F]}{\partial\theta}, \quad \frac{dl_i}{dt} = -\frac{\partial[F]}{\partial L_i}, \qquad (i=1, 2, \cdots, k). \tag{37}$$

Then

$$L_i = (L_i) + j_i\Theta, \qquad (i=1, 2, \cdots, k), \tag{38}$$

where the (L_i) are arbitrary constants. By means of these equations the linear variables L_i may be eliminated from the right members of all the differential equations (36) and (37), and thus be replaced by the single variable Θ. Thus the mentioned right members become functions only of the two variables Θ and θ; and, after Θ and θ are found in terms of t from the integration of (36), the L_i will be found from (38), and the l_i from a quadrature operated on the second equations of (37).

Then we can suppose that the integrals of (36) and (37) are written thus :

$$\left\{\begin{aligned}
\Theta &= \Theta_1\cos[\theta_0(t+c)] + \Theta_2\cos 2[\theta_0(t+c)] + \cdots, \\
L_1 &= (L_1) + j_1\Theta_1\cos[\theta_0(t+c)] + j_1\Theta_2\cos 2[\theta_0(t+c)] + \cdots, \\
&\cdots\cdots\cdots\cdots\cdots \\
L_k &= (L_k) + j_k\Theta_1\cos[\theta_0(t+c)] + j_k\Theta_2\cos 2[\theta_0(t+c)] + \cdots, \\
\theta &= \theta_0(t+c) + \theta_1\sin[\theta_0(t+c)] + \theta_2\sin 2[\theta_0(t+c)] + \cdots, \\
l_1 &= (l_1) + l_1^{(0)}(t+c) + l_1^{(1)}\sin[\theta_0(t+c)] + l_1^{(2)}\sin 2[\theta_0(t+c)] + \cdots, \\
&\cdots\cdots\cdots\cdots\cdots \\
l_k &= (l_k) + l_k^{(0)}(t+c) + l_k^{(1)}\sin[\theta_0(t+c)] + l_k^{(2)}\sin 2[\theta_0(t+c)] + \cdots,
\end{aligned}\right. \tag{39}$$

where the $(L_1), (L_2), \cdots, (L_k), (l_1), (l_2), \cdots, (l_k)$ are arbitrary constants. The latter set are not independent as they must satisfy the relation :

$$j_1(l_1) + j_2(l_2) + \cdots + j_k(l_k) = 0.$$

As there is the additional arbitrary constant c, the number of independent constants is $2k$ as it should be. The coefficients θ_0, $l_i^{(0)}$, $l_i^{(1)}$, $l_i^{(2)}$, $\cdots$, which may be regarded as functions of the (L_i), satisfy the relations :

$$\begin{aligned}
j_1 l_1^{(0)} + j_2 l_2^{(0)} + \cdots + j_k l_k^{(0)} &= \theta_0, \\
j_1 l_1^{(1)} + j_2 l_2^{(1)} + \cdots + j_k l_k^{(1)} &= \theta_1, \\
&\cdots\cdots
\end{aligned}$$

The formulas of transformation are then :

$$
(40)\quad \begin{cases}
\text{Replace} & L_1 & \text{by} & L_1 + j_1\Theta_1 \cos\theta + j_1\Theta_2 \cos 2\theta + \cdots, \\
 & \cdot\;\cdot\;\cdot & & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
\text{replace} & L_k & \text{by} & L_k + j_k\Theta_1 \cos\theta + j_k\Theta_2 \cos 2\theta + \cdots, \\
\text{replace} & l_1 & \text{by} & l_1 + l_1^{(1)} \sin\theta + l_1^{(2)} \sin 2\theta + \cdots, \\
 & \cdot\;\cdot\;\cdot & & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
\text{replace} & l_k & \text{by} & l_k + l_k^{(1)} \sin\theta + l_k^{(2)} \sin 2\theta + \cdots.
\end{cases}
$$

We have now to find what linear variables are conjugate to the new angular variables $l_1, l_2, \cdots l_k$. They are discovered immediately from a comparison of (34) and (38). As, from (34), it appears that Θ_0 the arbitrary constant, which may be conceived as annexed to the series for Θ, is not the element conjugate to the angular element $\theta_0(t + c)$, but that the expression

$$\tfrac{1}{2}[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots]$$

must be added to it to produce the required conjugate, it is plain from (38) that, after the transformation, the new L_i is no longer the exact conjugate of l_i, but that we have for that element the value

$$(41)\qquad L_i + \tfrac{1}{2}j_i[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots] \quad (i = 1, 2, \cdots, k).$$

VIII.

In making one of DELAUNAY'S transformations it is not absolutely necessary that we should employ the linear variables L_i, which are the conjugates of the angular variables l_i, in the development of the various series needed; we may use any others connected with the former by known relations. Then equation (41) will inform us, at any stage of the transformations, what function the conjugates of the angular variables are of the used linear variables. Thus, in making his developments in the lunar theory, DELAUNAY has not used the elements he calls L, G, H, but has substituted for them others which he names a, e, γ.

Let us suppose that the new set of linear elements we determine to use are denoted by the symbols $e_1, e_2, \cdots, e_k$. Then, in order to form the equivalents for the dl_i/dt, it will be necessary to know the values of the partial derivatives $\partial e_i/\partial L_j$ in terms of the e_i. The number of these derivatives is k^2, and we shall have as many equations for determining them. Having the L_i in terms of the e_i, the general form of these equations will be :

$$(42)\qquad \frac{\partial L_i}{\partial e_1}\frac{\partial e_1}{\partial L_j} + \frac{\partial L_i}{\partial e_2}\frac{\partial e_2}{\partial L_j} + \cdots + \frac{\partial L_i}{\partial e_k}\frac{\partial e_k}{\partial L_j} = 0 \text{ or } 1,$$

according as i and j are different or are the same. These equations divide themselves into k groups each containing k equations; each group serving to determine the k partial derivatives of the e_i with reference to one of the L_i which we denote by L_j. The functional determinant of each group of equations is the same, being the Jacobian of the variables $L_1, L_2, \cdots, L_k$ with reference to the variables $e_1, e_2, \cdots, e_k$. Calling this determinant Δ, we shall have

$$\frac{\partial e_i}{\partial L_j} = \frac{1}{\Delta} \frac{\partial \Delta}{\partial \left(\frac{\partial L_j}{\partial e_i}\right)}. \tag{43}$$

We shall much abbreviate the calculation of these partial derivatives by agreeing to compute $\partial e_i/\partial L_j$ not directly but by a correction to be added to the value it had before the transformation was made. This correction will be denoted by $D(\partial e_i/\partial L_j)$. Now the transformation being that made to remove from F all the periodic terms having the argument

$$j_1 l_1 + j_2 l_2 + \cdots + j_k l_k,$$

suppose that, in this transformation, we put

$$K = \tfrac{1}{2}[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \cdots], \tag{44}$$

and also

$$-K_m = \frac{\partial K}{\partial e_1}\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial K}{\partial e_2}\left(\frac{\partial e_2}{\partial L_m}\right) + \cdots + \frac{\partial K}{\partial e_k}\left(\frac{\partial e_k}{\partial L_m}\right), \tag{45}$$

where $(\partial e_i/\partial L_m)$ denotes the value of the partial derivative before the transformation; then we shall have the group of linear equations:

$$\left\{\begin{array}{l}
\frac{\partial L_1}{\partial e_1} D\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial L_1}{\partial e_2} D\left(\frac{\partial e_2}{\partial L_m}\right) + \cdots + \frac{\partial L_1}{\partial e_k} D\left(\frac{\partial e_k}{\partial L_m}\right) = j_1 K_m, \\
\frac{\partial L_2}{\partial e_1} D\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial L_2}{\partial e_2} D\left(\frac{\partial e_2}{\partial L_m}\right) + \cdots + \frac{\partial L_2}{\partial e_k} D\left(\frac{\partial e_k}{\partial L_m}\right) = j_2 K_m, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial L_k}{\partial e_1} D\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial L_k}{\partial e_2} D\left(\frac{\partial e_2}{\partial L_m}\right) + \cdots + \frac{\partial L_k}{\partial e_k} D\left(\frac{\partial e_k}{\partial L_m}\right) = j_k K_m.
\end{array}\right. \tag{46}$$

Put now

$$P_i = j_1 \frac{\partial \Delta}{\partial \left(\frac{\partial L_1}{\partial e_i}\right)} + j_2 \frac{\partial \Delta}{\partial \left(\frac{\partial L_2}{\partial e_i}\right)} + \cdots + j_k \frac{\partial \Delta}{\partial \left(\frac{\partial L_k}{\partial e_i}\right)}. \tag{47}$$

Then, evidently,

$$D\left(\frac{\partial e_i}{\partial L_m}\right) = \frac{P_i K_m}{\Delta}. \tag{48}$$

Thus the corrections for each partial derivative, caused by a given transformation, are reduced to the product of three factors, one of which is a function of i, another a function of m, and the third independent of either. The elements e_i, before the performance of any transformation, may be what we choose. DELAUNAY chose his a, e, γ so that they were connected with L, G, H, the conjugates of the angular variables l, g, h, by the relations:

$$L = \sqrt{\mu a}, \qquad G = L\sqrt{1-e^2}, \qquad H = G(1-2\gamma^2).$$

PART II.

APPLICATION OF DELAUNAY'S METHOD TO THE MINOR PLANET OF THE HECUBA TYPE.

IX.

DELAUNAY'S lunar theory affords a plentiful assortment of the transformations just discussed, but their application in a case of planetary motion gives rise to more complex expressions. In the lunar theory it is possible to expand all coefficients in power series of all the parameters involved; but, in a planetary theory where a, the ratio of the mean distances, is a considerable fraction, it is necessary to introduce the functions of a usually denoted by the symbol $b_s^{(i)}$, as also their derivatives with respect to a. It may therefore be profitable to give as simple an illustration as possible of these transformations where $b_s^{(i)}$ must be used.

Let Jupiter be supposed to describe a circular orbit about the Sun, while a small planet, without mass, describes an orbit in the same plane. Let the radius of Jupiter's orbit be taken as the linear unit; denote its longitude by $\epsilon' + n't$, and the masses severally of the Sun and Jupiter by m_0 and m'. Let a, e, l, and g be the mean distance, eccentricity, mean anomaly, and longitude of the perihelion of the small planet, and r and v its radius and true anomaly. Put

$$\gamma = g - \epsilon' - n't, \qquad L = \sqrt{m_0 a}, \qquad \Gamma = \sqrt{m_0 a(1-e^2)}.$$

The function we have denoted by F will have the expression:

$$(49) \quad F = \frac{m_0}{2a} + n'\Gamma + m'\{[1 - 2r\cos(v+\gamma) + r^2]^{-\frac{1}{2}} - r\cos(v+\gamma)\},$$

where r and v are to be eliminated through the equations:

$$r\cos v = a(\cos u - e), \quad r\sin v = a\sqrt{1-e^2}\sin u, \quad u - e\sin u = l.$$

The position of the small planet will be known when we know L, Γ, l, γ.

The differential equations for determining the latter are:

$$(50)\qquad \begin{cases} \dfrac{dL}{dt} = \dfrac{\partial F}{\partial l}, & \dfrac{dl}{dt} = -\dfrac{\partial F}{\partial L}, \\[2ex] \dfrac{d\Gamma}{dt} = \dfrac{\partial F}{\partial \gamma}, & \dfrac{d\gamma}{dt} = -\dfrac{\partial F}{\partial \Gamma}. \end{cases}$$

X.

In order to give an illustration of the transformations named by DELAUNAY *operations*, let us select from the periodic development of F, which, from (49), plainly has the form $\sum A_{i,i'} \cos(il + i'\gamma)$, all the terms having the argument $\theta = l + 2\gamma$. These will be terms of long period in case the small planet has a mean motion nearly double that of Jupiter, which case has been extensively discussed by astronomers, such a minor planet being called of the Hecuba type. Taking θ as one of the angular elements, we see that we can adopt γ as the other, and thus shall have $l = \theta - 2\gamma$. In order to obtain $[F]$ from F we have the equation:

$$(51)\qquad [F] = \frac{m_0}{2a} + n'\Gamma + \frac{1}{\pi}\int_0^\pi \frac{m'\,d\gamma}{\sqrt{1 - 2r\cos(v+\gamma) + r^2}},$$

remembering that r and v are now the same functions of $\theta - 2\gamma$ they were before of l. The last term of F in (49) is here omitted as it contributes nothing to $[F]$. Put

$$(52)\qquad \begin{aligned} [1 - 2r\cos(v+\gamma) + r^2]^{-\frac{1}{2}} = \tfrac{1}{2}B^{(0)} &+ B^{(1)}\cos(v+\gamma) \\ &+ B^{(2)}\cos 2(v+\gamma) + \cdots, \end{aligned}$$

where $B^{(i)}$ is the same function of r that $b_{\frac{1}{2}}^{(i)}$ is of a. In order that this series may be convergent it is necessary that $a(1+e) < 1$. Let us put

$$(53)\qquad A^{(i)} = \frac{1}{\pi}\int_0^\pi B^{(2i)} \cos i(2v - l)\, dl.$$

Then we have

$$(54)\qquad \begin{aligned} [F] = \frac{m_0}{2a} + n'\sqrt{m_0 a(1-e^2)} + m'\,[\tfrac{1}{2}A^{(0)} &+ A^{(1)}\cos\theta \\ &+ A^{(2)}\cos 2\theta + \cdots]. \end{aligned}$$

The investigation will be facilitated if we now make a slight change in the dependent variables employed so that they have the following equivalents:

$$(55)\qquad \begin{cases} \Theta = \sqrt{m_0 a}, & \Gamma = \sqrt{m_0 a}[2 - \sqrt{1 - e^2}], \\ \theta = l + 2g - 2\epsilon' - 2n't, & \gamma = \epsilon' + n't - g. \end{cases}$$

Then the differential equations determining the formulas of transformation are:

$$(56)\qquad \begin{cases} \dfrac{d\Theta}{dt} = \dfrac{\partial[F]}{\partial\theta}, & \dfrac{d\theta}{dt} = -\dfrac{\partial[F]}{\partial\Theta}, \\ \dfrac{d\Gamma}{dt} = 0, & \dfrac{d\gamma}{dt} = -\dfrac{\partial[F]}{\partial\Gamma}. \end{cases}$$

Of these equations the integral of the third, $\Gamma =$ a constant, furnishes the relation:

$$(57)\qquad a = \alpha[2 - \sqrt{1-e^2}]^{-2} = \alpha[1 - e^2 + \tfrac{1}{2}e^4 - \tfrac{1}{4}e^6 + \tfrac{3}{32}e^8 - \cdots],$$

α being a constant. By means of this relation the variable a may be eliminated from $[F]$ which thus will contain but two variables, e and θ. The equations (56) have the canonical form, but we prefer to discard the variable Γ and to use e in its stead. Supposing then that $[F]$ is made a function of the variables e and θ, the differential equations for the latter are:

$$(58)\qquad \begin{cases} \dfrac{de}{dt} = -\dfrac{1}{\sqrt{m_0 a}}\,\dfrac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]^2}{e}\,\dfrac{\partial[F]}{\partial\theta}, \\ \dfrac{d\theta}{dt} = \dfrac{1}{\sqrt{m_0 a}}\,\dfrac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]^2}{e}\,\dfrac{\partial[F]}{\partial e}, \end{cases}$$

where the factor

$$\frac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]^2}{e} = \frac{1}{e}[1 + \tfrac{1}{2}e^2 - \tfrac{1}{8}e^4 - \tfrac{3}{16}e^6 - \tfrac{17}{128}e^8 - \cdots].$$

These equations form a group to be integrated by themselves. After this integration is accomplished, γ is derived through a quadrature of the equation:

$$\frac{d\gamma}{dt} = -\frac{\partial[F]}{\partial\Gamma}.$$

In this equation $[F]$ is a function of Θ and Γ, but we have preferred to write it as a function of a and e; thus:

$$\frac{\partial[F]}{\partial\Gamma} = \frac{\partial[F]}{\partial a}\frac{\partial a}{\partial\Gamma} + \frac{\partial[F]}{\partial e}\frac{\partial e}{\partial\Gamma}.$$

But

$$\Theta = \frac{\sqrt{m_0 a}}{2-\sqrt{1-e^2}}, \qquad \Gamma = \sqrt{m_0 \alpha},$$

$$\alpha = \frac{\Gamma^2}{m_0}, \qquad \sqrt{1-e^2} = 2 - \frac{\Gamma}{\Theta}.$$

Consequently

$$\frac{\partial a}{\partial \Gamma} = 2\sqrt{\frac{a}{m_0}}, \qquad \frac{\partial e}{\partial \Gamma} = \frac{1}{\sqrt{m_0 a}} \frac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]}{e}.$$

Remembering that, with our adopted linear unit, $a' = 1$, $n' = \sqrt{m_0 + m'}$, we have:

$$\frac{1}{n'}\frac{d\gamma}{dt} = -\frac{1}{m_0\sqrt{\left(1+\dfrac{m'}{m_0}\right)a}}\left[2a\frac{\partial[F]}{\partial a} + \frac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]}{e}\frac{\partial[F]}{\partial e}\right].$$

But, adopting, as before, g for the longitude of the perihelion, this is more simply:

$$\frac{1}{n'}\frac{dg}{dt} = \frac{1}{\sqrt{\left(1+\dfrac{m'}{m_0}\right)a}}\left[2a\frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}[2-\sqrt{1-e^2}]}{e}\frac{\partial R}{\partial e}\right]$$

$$= \frac{2}{\sqrt{\left(1+\dfrac{m'}{m_0}\right)a}}\, a\frac{\partial R}{\partial a} + \frac{1}{2-\sqrt{1-e^2}}\frac{1}{n'}\frac{d\theta}{dt}$$

(59)

$$-\frac{1}{\sqrt{\left(1+\dfrac{m'}{m_0}\right)a}}[2-\sqrt{1-e^2}]^2 + \frac{2}{2-\sqrt{1-e^2}},$$

where

$$R = \frac{m'}{m_0}\frac{1}{\pi}\int_0^\pi \frac{d\gamma}{\sqrt{1-2r\cos(v-\gamma)+r^2}}.$$

XI.

In an application like the present, where the periodic developments of the various quantities are always tardily convergent, it is nearly impossible to give literal expressions for the coefficients. And, even if we consent to give to each coefficient its numerical value at once, the work of multiplying such periodic series together is very embarrassing, and the process easily leads to the commission of errors. Hence we adopt the method of substituting for each quantity involved the special values of it at equal intervals in the motion of the independent variable through the semicircumference. With this method of treatment it is necessary to separate the cases of non-libration and libration.

It is always an advantage in computation to have the members dealt with independent of any linear and temporal units. To this end let us substitute for the independent variable t, the variable $\tau = \epsilon' + n't$ or the longitude of Jupiter; also we put

$$W = \frac{[F]}{m_0}, \quad \nu = \frac{m'}{m_0}.$$

The coefficients of the periodic development of W are then absolute numbers. The equations which, with (57), we shall use for the elaboration of the problem, are the three following equations:

$$(60)\quad \begin{cases} W = \text{a constant}, \\ \dfrac{d\tau}{d\theta} = \sqrt{(1+\nu)\,a}\,\dfrac{e}{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]^2}\,\dfrac{1}{\dfrac{\partial W}{\partial e}}, \\ \dfrac{dg}{d\tau} = \dfrac{2}{\sqrt{(1+\nu)\,a}}\,a\,\dfrac{\partial R}{\partial a} + \dfrac{1}{2-\sqrt{1-e^2}}\,\dfrac{d\theta}{d\tau} \\ \qquad - \dfrac{1}{\sqrt{(1+\nu)\,a}}\,[2-\sqrt{1-e^2}]^2 + \dfrac{2}{2-\sqrt{1-e^2}}; \end{cases}$$

W has the expression:

$$W = \frac{1}{2a}\,[2-\sqrt{1-e^2}]^2 + \sqrt{(1+\nu)\,a}\,\frac{1-e^2+2\sqrt{1-e^2}}{3+e^2} + \nu\frac{1}{\pi}\int_0^\pi \frac{d\gamma}{\Delta}$$

$$(61)\quad = \frac{1}{2a}\,[1 + e^2 + \tfrac{1}{2}e^4 + \tfrac{1}{4}e^6 + \tfrac{5}{32}e^8 + \tfrac{7}{64}e^{10} + \cdots]$$

$$+ \sqrt{(1+\nu)\,a}\,[1 - e^2 + \tfrac{1}{4}e^4 - \tfrac{1}{8}e^6 + \tfrac{1}{64}e^8 - \tfrac{3}{128}e^{10} + \cdots] + R.$$

This equation contains as variables only e and θ; hence, since e should never be negative, the dependence of the two variables on each other may be shown graphically by taking e as the radius and θ as the angle in a system of polar coördinates. If we are given a pair of simultaneous values of e and θ, it is obvious that by their aid we can determine the constant value of W. Desiring to ascertain at what points on the axis the curve passes we make in (61) in succession $\theta = 0°$ and $\theta = 180°$ and we get two equations of the forms:

$$(62)\quad \begin{cases} D = M_1 e + M_2 e^2 + M_3 e^3 + M_4 e^4 + M_5 e^5 + \cdots, \\ D = -M_1 e + M_2 e^2 - M_3 e^3 + M_4 e^4 - M_5 e^5 + \cdots, \end{cases}$$

where D may be regarded as the arbitrary constant and the M are constants, being functions of a and ν. These equations are transcendental in e and are such that the positive roots of the one are equivalent to the negative roots of the other. If each has a positive real root continuous with the value of e which was used for the determination of the constant value of W, the variable θ generally moves through the whole range of real values. But, if the first equation has two positive real roots and the second none, θ will librate about the value $\theta = 0^\circ$. But, if the second has two positive real roots and the first none, θ will librate about $\theta = 180^\circ$. It will be seen that when $D = 0$ we have the limit separating non-libration from libration.

XI.

Case I.—*Non-libration.*—Here, as θ goes through the semicircumference, it can be employed as the independent variable. Then, in the first equation of (60), we assign to θ, in succession, a series of equidistant values covering the semicircumference. (Those used in our illustrative examples are 13 in number, viz, $\theta = 0^\circ$, $\theta = 15^\circ$, $\theta = 30^\circ$, $\cdots$, $\theta = 180^\circ$.) This procedure furnishes us with a like number of equations for determining the corresponding values of e. Solving these by the tentative process we have these values of e, and can apply to them the procedure of mechanical quadratures. Thus is obtained a general expression for e as a periodic function of θ involving only cosines.

As the next step these special values of e can be substituted in the right member of the second equation of (60). To the special values thus obtained for $d\tau/d\theta$ can be applied mechanical quadratures, and the resulting periodic series, involving only cosines of integral multiples of θ, can be integrated with respect to this variable. This integral may be put in the form:

$$(63) \qquad \theta_0(t + c) = \theta + \beta_1 \sin\theta + \beta_2 \sin 2\theta + \beta_3 \sin 3\theta + \cdots.$$

Knowing θ_0 we are now in possession of the period of the inequalities we are endeavoring to derive. The left member of this equation we shall designate as the time-argument, and, for brevity, denote it as ζ. In the next place we assign to ζ a series of equidistant values going from 0° to 180°, and, by a tentative process applied to (63), arrive at the corresponding values of θ. These corresponding values of θ can be substituted for θ in the expression of e as a periodic function of θ, and thus we shall have the values of e which correspond to the equidistant values of ζ. We can now readily derive the similar values of the two quantities $e\cos\theta$ and $e\sin\theta$. To these we apply mechanical quadratures and thus obtain the periodic developments of these quantities in terms of ζ.

As the last step in this work we can, through the last equation of (60), express $dg/d\tau$ as a function of e and θ, and, by the substitution of the special values of the latter variables, obtain the special values of $dg/d\tau$ which correspond to the equidistant values of ζ. To these apply mechanical quadratures and the periodic series for $dg/d\tau$ is obtained. This being integrated we have the series for g, and the solution of the problem is completed.

XII.

Case II.—*Libration.*—Here we are cut off from the use of θ as an independent variable on account of its not going through the semicircumference. But this difficulty is surmounted by substituting for it another variable which does move continuously from $-\infty$ to $+\infty$. In order to ascertain, in the case of libration, the limiting values of θ we have to solve the simultaneous equations:

$$W = a \text{ constant}, \quad \frac{\partial W}{\partial e} = 0,$$

the unknowns being e and θ. That is to say, a value of θ must be found which will make the first equation have two equal roots for e. This can be done by a tentative process. If we assume θ too large, generally, we shall not be able to discover real values for e from the first equation; but, if θ is taken too small, we get two values real but unequal for e. These two conditions must be brought as close as possible until we discover the point of passage from one to the other. In our illustrative example we escape the necessity of this tentative process by assuming as one of the two fundamental elements of the example not the D of (62) but the amount of libration.

The amount of libration being thus either assumed or determined, let κ denote the limiting value of $\sin\theta$; we then can put

$$\sin\theta = \kappa \sin\psi; \tag{64}$$

and the motion of ψ can be regarded as extending continuously from $-\infty$ to $+\infty$. Adopting the variable ψ for replacing θ, the second equation of (60) takes the form:

$$\frac{d\tau}{d\psi} = \sqrt{(1+\nu)a}\,\frac{e}{\sqrt{1-e^2}[2-\sqrt{1-e^2}]^2}\,\frac{\kappa\cos\psi}{\sqrt{1-\kappa^2\sin^2\psi}}\,\frac{1}{\frac{\partial W}{\partial e}}, \tag{65}$$

where the newly introduced radical must receive the sign of $\cos\theta$. We can now make ψ play the same rôle as θ did in Case I, and there is need of no further explanations.

XIII.

We attend now to the integration of equations (60). *The operation of Delaunay's lunar theory which is numbered 23* has great affinity with that here detailed, and the two may be compared.* He, it is true, has six variables to our four; but, in comparing, his γ should be made to vanish and his h then becomes indeterminate.

* Mémoires de l'Académie des Sciences, vol. XXVIII, p. 493.

The periodic development of the reciprocal of the distance between two planets as a function of the time has been given by LEVERRIER to terms of the seventh order inclusive, and those of the eighth order have afterwards been added by M. BOUQUET.* We avail ourselves of this development and adopt the mode of LEVERRIER for noting the coefficients except in the portion which is a function of e alone. We put $A_j^{(i)} = (1/j!)\, a^j\, d^j b_{\frac{1}{2}}^{(i)}/da^j$, $j = 0$ in the portion factored by $\cos 0\theta$, $j = 2$ in the portion factored by $\cos\theta$, $j = 4$ in the portion factored by $\cos 2\theta$ and so on; only the numerical factors are written since the A can easily be filled in as they always commence with $A_0^{(i)}$, and the lower index always increases by a unit in each step to the right. With LEVERRIER we put χ for $\frac{1}{2}e$. This then is the development of a'/Δ, preserving only the terms involving the integral multiples of θ as arguments:

$$
\begin{aligned}
\frac{a'}{\Delta} = {} & \frac{1}{2}A_0^{(0)} + [A_1^{(0)} + A_2^{(0)}]\chi^2 + \frac{3}{1}[A_3^{(0)} + A_4^{(0)}]\chi^4 + \frac{4.5}{1.2}[A_5^{(0)} + A_6^{(0)}]\chi^6 + \frac{5.6.7}{1.2.3}[A_7^{(0)} + A_8^{(0)}]\chi^8 \\
& + \Big\{ -[4+1]\chi + \Big[14 + \frac{5}{2} - 6 - 3\Big]\chi^3 + \Big[-\frac{5}{3} - \frac{53}{12} + \frac{34}{3} + 5 - 16 - 10\Big]\chi^5 \\
& \qquad + \Big[\frac{271}{36} - \frac{203}{144} + \frac{19}{4} - \frac{49}{8} + 20 + \frac{25}{2} - 50 - 35\Big]\chi^7 \Big\} \cos\theta \\
& + \Big\{ [22+7+1]\chi^2 + \Big[-\frac{596}{3} - \frac{212}{3} + 8 + 16 + 4\Big]\chi^4 + \Big[\frac{1300}{3} + \frac{743}{3} - 49 \\
& \qquad - 112 + 2 + 45 + 15\Big]\chi^6 + \Big[-\frac{8312}{45} - \frac{500212}{45} - \frac{280}{9} + \frac{1228}{5} \\
& \qquad + 0 - 236 - 16 + 140 + 56\Big]\chi^8 \Big\} \cos 2\theta \\
& + \Big\{ -\Big[134 + \frac{93}{2} + 10 + 1\Big]\chi^3 + \Big[\frac{4053}{2} + \frac{6289}{8} + 107 - \frac{117}{2} - 32 - 5\Big]\chi^5 \\
& \qquad + \Big[-\frac{64177}{40} - \frac{384789}{80} - \frac{12123}{20} + \frac{6249}{8} + 348 - \frac{189}{2} - 102 - 21\Big]\chi^7 \Big\} \cos 3\theta \\
& + \Big\{ \Big[\frac{2570}{3} + \frac{932}{3} + 80 + 13 + 1\Big]\chi^4 + \Big[-\frac{275528}{15} - \frac{109972}{15} - 1676 + 48 + 176 \\
& \qquad + 54 + 6\Big]\chi^6 + \Big[\frac{6259444}{45} + \frac{592976}{9} + \frac{234338}{15} - \frac{44756}{15} - \frac{10036}{3} - \frac{1432}{3} \\
& \qquad + 400 + 196 + 28\Big]\chi^8 \Big\} \cos 4\theta \\
& + \Big\{ -\Big[\frac{33797}{6} + \frac{50345}{24} + \frac{1795}{3} + \frac{245}{2} + 16 + 1\Big]\chi^5 + \Big[\frac{5652235}{36} + \frac{9141589}{144} + \frac{210217}{12} \\
& \qquad + \frac{48985}{24} - \frac{2020}{3} - \frac{775}{2} - 82 - 7\Big]\chi^7 \Big\} \cos 5\theta \\
& + \Big\{ \Big[\frac{188616}{5} + \frac{71499}{5} + 4357 + 1024 + 174 + 19 + 1\Big]\chi^6 + \Big[-\frac{45378432}{35} \\
& \qquad - \frac{3695460}{7} - \frac{803616}{5} - \frac{150828}{5} + 80 + 2156 + 720 + 116 + 8\Big]\chi^8 \Big\} \cos 6\theta
\end{aligned}
\tag{66}
$$

---

* Annales de l'Observatoire de Paris, vols. I, XIX.

$$+\left\{-\frac{46064791}{2520}+\frac{70738549}{720}+\frac{1880921}{60}+\frac{193921}{24}+\frac{4844}{3}+\frac{469}{2}-22-1\right\}\chi^7\cos 7\theta$$

$$+\left\{\frac{552146674}{315}+\frac{213998824}{315}+\frac{2018552}{9}+\frac{926516}{15}+\frac{41380}{3}+\frac{7192}{3}+304+25+1\right\}\chi^8\cos 8\theta\,.$$

From this expression we must eliminate the variable a by means of its value in terms of e given by (57). We put

$$p=[2-\sqrt{1-e^2}]^{-2}-1\,.$$

Let LEVERRIER'S coefficient of $\cos j\theta$ be denoted thus:

$$c_0A_0^{(2j)}+c_1A_1^{(2j)}+c_2A_2^{(2j)}+\cdots=\sum c_iA_i^{(2j)}\,,$$

where c_i is a function of e. Denoting the similar coefficient, after the variable a has been eliminated through (57) by $\sum f_iA_i^{(2j)}$, we evidently have

$$f_i=(1+p)^ic_i+\frac{i}{1}(1+p)^{i-1}pc_{i-1}+\frac{i(i-1)}{1.2}(1+p)^{i-2}p^2c_{i-2}+\cdots.$$

By means of this formula we obtain the following expression, in which a, the argument of the various quantities $A_i^{(2j)}$, is the constant a of (57):

$$\frac{a'}{\Delta}=\frac{1}{2}+[0-1+1]\chi^2+[0+0-8-9+3]\chi^4+[0+0+48+100+0-50+10]\chi^6$$
$$+[0-4-160-600-224+720+240-245+35]\chi^8$$
$$+\left\{-[4+1]\chi+\left[14+\frac{45}{2}+2-3\right]\chi^3+\left[-\frac{5}{3}-\frac{1325}{12}-\frac{218}{3}+65+32-10\right]\chi^5\right.$$
$$\left.+\left[\frac{271}{36}+\frac{33829}{144}+\frac{6209}{12}-\frac{3025}{8}-796+\frac{105}{2}+190-35\right]\chi^7\right\}\cos\theta$$
$$+\left\{[22+7+1]\chi^2+\left[-\frac{596}{3}-\frac{560}{3}-56+4+4\right]\chi^4+\left[\frac{1300}{3}+1557+\frac{3517}{3}\right.\right.$$
$$\left.+248-222-35+15\right]\chi^6+\left[-\frac{8312}{45}-\frac{740632}{45}-\frac{95032}{9}-\frac{19112}{5}\right.$$
$$\left.\left.+3680+2184-496-280+56\right]\chi^8\right\}\cos 2\theta$$
$$+\left\{-\left[134+\frac{93}{2}+10+1\right]\chi^3+\left[\frac{4053}{2}+\frac{12065}{8}+559+\frac{147}{2}-16-5\right]\chi^5\right.$$
$$\left.+\left[-\frac{64177}{40}-\frac{1400349}{80}-\frac{248943}{20}-\frac{26439}{8}+612+\frac{971}{2}+18-21\right]\chi^7\right\}\cos 3\theta$$
(67)
$$+\left\{\left[\frac{2570}{3}+\frac{932}{3}+80+13+1\right]\chi^4+\left[-\frac{275528}{15}-\frac{60004}{5}-\frac{14404}{3}-1068-48\right.\right.$$
$$\left.+34+6\right]\chi^6+\left[\frac{6259444}{45}+\frac{1602224}{9}+\frac{1180994}{15}+\frac{630004}{15}\right.$$
$$\left.\left.+\frac{11372}{3}-\frac{7912}{2}-800+28+28\right]\chi^8\right\}\cos 4\theta$$

$$+\left\{-\left[\frac{33797}{6}+\frac{50345}{24}+\frac{1795}{3}+\frac{245}{2}+16+1\right]\chi^5+\left[\frac{5652235}{36}+\frac{13594381}{144}\right.\right.$$

$$\left.\left.+\frac{469037}{12}+\frac{256585}{24}+\frac{4628}{3}-\frac{95}{2}-58-7\right]\chi^7\right\}\cos 5\theta$$

$$+\left\{\left[\frac{188616}{5}+\frac{71499}{5}+4357+1024+174+19+1\right]\chi^6+\left[-\frac{45378432}{35}\right.\right.$$

$$\left.\left.-\frac{5152104}{7}-\frac{1549888}{5}-\frac{473688}{5}-19088-1704+240+88+8\right]\chi^8\right\}\cos 6\theta$$

$$+\left\{-\frac{46064791}{2520}+\frac{70738549}{720}+\frac{1880921}{60}+\frac{191863}{24}+\frac{4844}{3}+\frac{469}{2}+21+1\right\}\chi^7\cos 7\theta$$

$$+\left\{\frac{552146674}{315}+\frac{213998824}{315}+\frac{2018552}{9}+\frac{926516}{15}+\frac{41380}{3}+\frac{7192}{3}+304+25+1\right\}\chi^8\cos 8\theta .$$

In forming the value of $dg/d\tau$ we need to know the derivative of the foregoing expression with respect to a. By noting the equation:

$$a\frac{\partial A_i^{(2j)}}{\partial a}=iA_i^{(2j)}+(i+1)A_{i+1}^{(2j)} ,$$

and changing our mode of noting the coefficients so that the number first given is the coefficient of $A_1^{(2j)}$ instead of $A_0^{(2j)}$, we have:

$$a\frac{\partial \frac{a'}{\Delta}}{\partial a}=\tfrac{1}{2}+[-1+0+3]\chi^2+[0-16-51-24+15]\chi^4+[0+96+444+400-250$$

$$-240+70]\chi^6+[-4-328-2280-3296+2480+5760-35-1680+315]\chi^8$$

$$+\left\{-[5+2]\chi+\left[\frac{73}{2}+49-3-12\right]\chi^3+\left[-\frac{1345}{12}-\frac{2197}{6}-23+388+110-60\right]\chi^5\right.$$

$$\left.+\left[\frac{34913}{144}+\frac{108337}{72}+\frac{3343}{8}-\frac{9093}{2}-\frac{7435}{2}+1455+1085-280\right]\chi^7\right\}\cos\theta$$

$$+\left\{[29+16+3]\chi^2+\left[-\frac{1156}{3}-\frac{1456}{3}-156+32+20\right]\chi^4+\left[\frac{5971}{3}\right.\right.$$

$$\left.+\frac{16376}{3}+4261+104-1285-120+105\right]\chi^6+\left[-\frac{83216}{5}-\frac{270176}{5}\right.$$

$$\left.\left.-\frac{647168}{15}-\frac{2848}{5}+29320+10128-5432-1792+504\right]\chi^8\right\}\cos 2\theta$$

$$+\left\{-\left[\frac{361}{2}+113+33+4\right]\chi^3+\left[\frac{28277}{8}+\frac{16537}{4}+\frac{3795}{2}+230-105-30\right]\chi^5\right.$$

$$+\left[-\frac{1528703}{80}-\frac{2396121}{40}-\frac{1890243}{40}-\frac{21543}{2}+\frac{11975}{2}\right.$$

$$\left.\left.+3021-21-168\right]\chi^7\right\}\cos 3\theta$$

(68)

$$+\left\{\left[\frac{3502}{3}+\frac{2344}{3}+279+56+5\right]\chi^4+\left[-\frac{91108}{3}-\frac{504064}{15}-17608-4464\right.\right.$$

$$\left.-70+240+42\right]\chi^6+\left[\frac{14270564}{45}+\frac{23108204}{45}+\frac{1810998}{5}+\frac{2747456}{15}\right.$$

$$\left.\left.+\frac{17300}{5}-20624-5404+448+252\right]\chi^8\right\}\cos 4\theta$$

$$+\Big\{-\Big[\frac{185533}{24}+\frac{64705}{12}+\frac{4325}{2}+554+85+6\Big]\chi^5+\Big[\frac{36203321}{144}+\frac{19222825}{72}$$
$$+\frac{1194659}{8}+\frac{293609}{6}+\frac{44855}{6}-633-455-56\Big]\chi^7\Big\}\cos 5\theta$$
$$+\Big\{\Big[52023+\frac{186568}{5}+16143+4792+965+120+7\Big]\chi^6$$
$$+\Big[-\frac{71138952}{35}-\frac{73219472}{35}-\frac{6070728}{5}-\frac{2276512}{5}-103960-8784$$
$$+2296+768+72\Big]\chi^8\Big\}\cos 6\theta$$
$$-\Big\{\frac{587299425}{5040}+\frac{93309601}{360}+\frac{4721157}{40}+\frac{230615}{6}+\frac{55475}{6}+1539+161+8\Big\}\chi^7\cos 7\theta$$
$$+\Big\{\frac{766145498}{315}+\frac{569296288}{315}+\frac{12872308}{15}+\frac{4533664}{15}+\frac{242860}{3}+16208$$
$$+2303+208+9\Big\}\chi^8\cos 8\theta.$$

It is desirable to have the means of verifying these truncated developments of a'/Δ derived from the work of LEVERRIER and M. BOUQUET. In fact, by the application of the first of two following theorems, an error has been found in M. BOUQUET'S expression for (225); in the coefficient of K_3, $-h$ should be substituted for h. The two theorems are the following:

The coefficient of $\cos j\theta$ *in the periodic development of* a'/Δ *is the same as that of* s^j *in the expansion of the expression*

$$\sum_{i=0}^{i=\infty} A_i^{(2j)}(-\chi)^i\left(s+\frac{1}{s}\right)^i\left[1-\chi\left(s+\frac{1}{s}\right)\right]\left(\frac{s-\omega}{1-\omega s}\right)^{2j}\epsilon^{j\chi\left(s-\frac{1}{s}\right)}$$

in a power series with reference to s.

The coefficient of $\cos j\theta$ *in the same development after* a *has been replaced by* $a[2-\sqrt{1-e^2}]^2$ *is the same as that of* s^j *in the expansion of the expression*

$$\sum_{i=0}^{i=\infty} A_i^{(2j)}\left[p-\chi(1+p)\left(s+\frac{1}{s}\right)\right]^i\left[1-\chi\left(s+\frac{1}{s}\right)\right]\left(\frac{s-\omega}{1-\omega s}\right)^{2j}\epsilon^{j\chi\left(s-\frac{1}{s}\right)}$$

in a power series with reference to s.

In these expressions ω stands for $e/(1+\sqrt{1-e^2})$.

XIV.

The two linear elements which determine all the coefficients in the periodic developments involved in this problem may be taken to be the constant a of (57) and the constant D of (62). It is proposed to elaborate two examples illustrating the subject in hand, one exhibiting non-libration, the other libration. In both we will assign to a such a value as makes $\log a = 9.8$. This value makes the period of revolution of the small planet nearly or exactly half that of

Jupiter. Whether we are to have a case of non-libration or libration will then depend on the value assigned to the second constant D.

In the first place then we compute the values of such of the quantities $A_i^{(j)}$ as are needed in this investigation, correspondent to $\log a = 9.8$, by procedures which it is unnecessary to detail. The results are contained in the following table:

Values of log $A_i^{(j)}$ for log $a = 9.8$.

j	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$
0	0.354 4041 774	9.845 4797 897	9.935 0116 655	9.989 1230	0.111 3716
2	9.564 3962 993	9.965 8367 1	0.002 7463 5	0.007 7852	0.121 2342
4	9.035 0709 047	9.694 9897 1	9.992 6030 6	0.098 8192	0.164 1565
6	8.555 0516 205	9.374 1611 5	9.842 7442 0	0.099 5969	0.231 3986
8	8.096 8549 86	9.031 7132 5	9.624 7440 8	0.005 5654	0.239 5776
10	7.651 0634 5	8.677 0326 9	9.367 4385 2	9.849 2650	0.178 8047
12	7.213 2942 8	8.314 4461 7	9.084 6837 2	9.649 5418	0.062 8948
14	6.781 1495	7.946 3186 8	8.784 1408 8	9.419 6593	9.905 2677
16	6.353 1544	7.574 0873	8.470 4851	9.167 1668	9.715 8220

j	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
0	0.251 2555	0.408 5603	0.576 748	0.753 37	0.937 0
2	0.257 4382	0.412 7086	0.579 698	0.755 57	0.938 5
4	0.278 2372	0.426 0622	0.588 970	0.762 42	0.943 9
6	0.329 4253	0.453 2441	0.606 143	0.774 66	0.956 8
8	0.385 1308	0.503 3035	0.637 049	0.794 30	0.977 1
10	0.398 9502	0.552 5313	0.684 255	0.826 57	1.004 5
12	0.360 4400	0.570 3163	0.729 206	0.870 86	1.029 0
14	0.272 3635	0.545 6897	0.750 571	0.909 38	1.045 5
16	0.146 2268	0.480 0088	0.721 959	0.935 78	1.045 7

Substituting these values in (67) we get

$$
\begin{aligned}
\frac{a'}{\Delta} = & \quad 1.1307697497 + 0.04010033e^2 - 0.7367846e^4 + 1.17661e^6 + 0.6155e^8 \\
& + [-1.19571949 + 3.1113902\,e^2 - 2.669146\,e^4 - 1.90033e^6]e \cos\theta \\
& + [1.70905245 - 9.8883917\,e^2 + 30.18579\,e^4 - 62.2057e^6]e^2 \cos 2\theta \\
& + [-3.00445698 + 27.190861\,e^2 - 117.01214\,e^4]e^3 \cos 3\theta \\
& + [5.7966694 - 71.99282\,e^2 + 369.2943\,e^4]e^4 \cos 4\theta \\
& + [-11.800399 + 186.12652\,e^2]e^5 \cos 5\theta \\
& + [24.86635 - 475.7506\,e^2]e^6 \cos 6\theta \\
& - 52.40299e^7 \cos 7\theta \\
& + 118.0918\,e^8 \cos 8\theta\,.
\end{aligned}
\tag{69}
$$

We adopt the mass of Jupiter so that $\nu = 1/1047.355$. Then, in the expression (61) of W, the portion which is independent of the interaction of Jupiter and the small planet, developed in powers of e^2, becomes:

$$
\begin{aligned}
& 1.58715\,39467\,862 \\
& -0.00226\,07543\,2\quad e^2 \\
& +0.59490\,01358\quad e^4 \\
& +0.09877\,323\quad e^6 \\
& +0.13623\,71\quad e^8 \\
& +0.06804\,8\quad e^{10}\,.
\end{aligned}
$$

If we omit from the expansion of W its constant term, and call D the constant of the thus modified W, as in (62), we have, as an integral of our problem,

$$
(70)\qquad
\begin{aligned}
D = & -0.00222246709e^2 + 0.5941966641e^4 + 0.09989664\,e^6 + 0.1368248e^8 \\
& + [-0.00114165636 + 0.0029707121e^2 - 0.002548463e^4 - 0.00181441e^6]\,e \cos\theta \\
& + \ [0.00163177054 - 0.0094412989e^2 + 0.02882097\,e^4 - 0.593931e^6]\,e^2 \cos 2\theta \\
& + [-0.0028686138 + 0.025961456\,e^2 - 0.11172156\,e^4]\,e^3 \cos 3\theta \\
& + \ [0.0055345794 - 0.06873774\,e^2 + 0.3525971\,e^4]\,e^4 \cos 4\theta \\
& + [-0.01126686 + 0.1777110e^2]\,e^5 \cos 5\theta \\
& + \ [0.02374204 - 0.4542400e^2]\,e^6 \cos 6\theta \\
& - \ 0.0500337e^7 \cos 7\theta \\
& + \ 0.1127524e^8 \cos 8\theta\,.
\end{aligned}
$$

By making $\theta = 0°$ in the preceding equation, we get, as the correspondent of the first equation of (62), the following:

$$
(71)\qquad
\left\{
\begin{array}{ll}
-0.00114\,16563\,6\,e & -0.00059\,06875\,5\,e^2 \\
+0.00010\,20983\,5\,e^3 & +0.59028\,99445\quad e^4 \\
+0.01214\,61357\quad e^5 & +0.08372\,1913\quad e^6 \\
+0.01414\,140\quad e^7 & +0.08854\,10\quad e^8
\end{array}
\right\} = D\,.
$$

It will be seen by comparison of the coefficients of this equation that, unless e is very small, it will not do to regard the equation as approximately a quadratic in e; for $e = 0.1$ the term in e^4 is ten times more important than the term in e^2. The supposition that the mean motion of the small planet is nearly double that of Jupiter makes the coefficient of e^2 nearly vanish. In fact a very small change in the adopted value of a would make this coefficient 0.

What sort of a curve we shall have exhibiting graphically the connection between e and θ will depend on the value assigned to D. To bring this out in a clear manner we compute the values of the left member of the preceding equation for each 0.01 in the value of e between the limits $\pm$ 0.3, and thus have the following table:

e	D	e	D	e	D	e	D
−0.30	+0.0051 0216	−0.15	+0.0004 5648	+0.01	−0.0000 1148	+0.16	+0.0001 9224
0.29	44 8080	0.14	3 7472	0.02	2298	0.17	2 8623
0.28	39 2021	0.13	3 0675	0.03	3430	0.18	4 0095
0.27	34 1625	0.12	2 5066	0.04	4509	0.19	5 3896
0.26	29 6473	0.11	2 0477	0.05	5485	0.20	7 0298
0.25	25 6178	0.10	1 6715	0.06	6294	0.21	8 9590
0.24	22 0365	0.09	1 3659	0.07	6857	0.22	11 2071
0.23	18 8675	0.08	1 1166	0.08	7082	0.23	13 8067
0.22	16 0764	0.07	9115	0.09	6861	0.24	16 7912
0.21	13 6306	0.06	7400	0.10	6073	0.25	20 1960
0.20	11 4987	0.05	5928	0.11	4582	0.26	24 0580
0.19	9 6512	0.04	4622	0.12	−0.0000 2236	0.27	28 4159
0.18	8 0599	0.03	3420	0.13	+0.0000 1129	0.28	33 3102
0.17	6 6982	0.02	2269	0.14	5695	0.29	38 7834
−0.16	+0.0005 5411	−0.01	+0.0000 1136	+0.15	+0.0001 1657	+0.30	+0.0044 8802

As e ought always to be positive, in the first half of this table we may change the sign of e, provided we suppose that the corresponding value of D is regarded as appertaining to the special value 180° for θ, while, in the remainder of the table, this value corresponds to $\theta = 0°$.

From the course of the values of D in the table we see there is one minimum $= -0.00007\ 082$, which occurs for $e = 0.08$ about; consequently, if D is chosen greater than this the equation (71) will have two real roots for e. If D is positive one of these roots will be negative; changing the sign of the latter it will belong to the value $\theta = 180°$; the positive root will belong to $\theta = 0°$. Thus, in this case, the motion of θ is generally through the whole semicircumference, and hence is continuous from $-\infty$ to $+\infty$. But, if D is negative, both roots will be positive, and thus belong to $\theta = 0°$. In this case, therefore, θ departs from 0° and comes back to it without having reached 180°. This is called a libration; we see that $D = 0$ marks the dividing point between continuous and libratory motion for θ. The latter case also has the largest swing in the values of e, viz, from $e = 0$ to about $e = 0.127$. Generally, the larger D is, the smaller will be the variation in e. Thus, if $D = +0.0045$, e will vary from 0.29 to 0.30. If there is libration e cannot exceed 0.127. These remarks, however, must be understood as applying only to the values holding for $\theta = 0°$ and $\theta = 180°$. Larger values for e may obtain for values of θ lying between 0° and 180°.

XV.

For our illustrative example, in the case of a continuous motion for θ, we assign to D the value $+0.0001$ in (70). All the coefficients of the various periodic series will now have determinate numerical values. The preceding table shows that, for this assumption, the eccentricity will have, when $\theta = 0°$, the approximate value $e = 0.1475$, and, when $\theta = 180°$, the approximate value $e = 0.0745$. In this case these are the limiting values, as e continuously diminishes while θ is passing from 0° to 180°.

Attending now to the elaboration of our selected example, in (70) we give to θ, in succession, the values 15°, 30°, 45°, 60°, 75°, 90° and get as the right member of (70), [the value $\theta = 0°$ has already been considered in (70)],

θ				
15°	$-0.00110\ 27554e$	$-0.00080\ 93046e^2$	$+0.00084\ 1072e^3$	$+0.58878\ 755e^4$
30	98 87034	140 65774	257 2712	58670 872
45	80 72730	222 24671	412 9027	58866 208
60	57 08282	303 83568	435 3970	59615 002
75	$-0.00029\ 54824$	363 56296	$+0.00279\ 7293$	60514 036
90	0	$-0.00385\ 42466$	0	$+0.60917\ 254$

θ				
15°	$+0.01297\ 98e^5$	$+0.09048\ 7e^6$	$-0.02180\ 7e^7$	$+0.20531e^8$
30	$+0\ 00755\ 03$	$+0.12493\ 4$	$-0.11214\ 3$	$+0.32869$
45	$-0.01219\ 27$	$+0.16863\ 4$	$-0.08332\ 4$	-0.10302
60	$-0.03286\ 91$	$+0.14359\ 7$	$+0.17465\ 3$	-0.52039
75	$-0.02990\ 01$	$+0.04056\ 8$	$+0.29851\ 4$	$+0.30818$
90	0	$-0.02140\ 4$	0	$+1.11581$

The coefficients in the second quadrant for θ are the same as in the first, but in reverse order, except, that for the odd powers of e the sign must be reversed.

Making the right members of these 13 equations equal to $+ 0.0001$, we solve them with reference to e as the unknown, and substitute, in succession, the values thus obtained and the corresponding value of θ in the right member of the second equation of (60). The results obtained are the following:

θ	e	$d\tau/d\theta$
0°	0.14746 2372	18.321384
15	0.14702 7366	18.443969
30	0.14569 2650	18.827993
45	0.14335 8947	19.521595
60	0.13982 8670	20.600475
75	0.13480 9503	22.197678
90	0.12791 2165	24.508178
105	0.11869 0066	27.615097
120	0.10709 4867	30.992715
135	0.09444 8152	32.866681
150	0.08360 2031	31.947061
165	0.07679 7488	29.983003
180	0.07454 7767	29.112462

The mean of the numbers in the third column, attributing half weight to the first and last, is 25.101781; and this is the number of revolutions of Jupiter in the period of the inequalities we are investigating. If the sidereal revolution of Jupiter is put at 11.861980 Julian years, the latter period is 297.75681 such years.

From the special values given in this table we can derive the two periodic series representing them. Integrating the latter, and, for brevity, putting ζ for $\theta_0(t + c)$, we get the following expressions:

$$e = \left\{\begin{array}{l} 0.11918\ 891 \\ +0.03553\ 171 \cos \theta \\ -0.00857\ 010 \cos 2\theta \\ +0.00123\ 337 \cos 3\theta \\ +0.00027\ 721 \cos 4\theta \\ -0.00029\ 767 \cos 5\theta \\ +0.00012\ 029 \cos 6\theta \\ -0.00001\ 828 \cos 7\theta \\ -0.00000\ 783 \cos 8\theta \\ +0.00000\ 780 \cos 9\theta \\ -0.00000\ 374 \cos 10\theta \\ +0.00000\ 037 \cos 11\theta \\ +0.00000\ 033 \cos 12\theta \end{array}\right\}, \qquad \zeta = \theta + \left\{\begin{array}{l} -60173''.40 \sin \theta \\ -\ 1643.74 \sin 2\theta \\ +\ 4612.77 \sin 3\theta \\ -\ 2132.15 \sin 4\theta \\ +\ 544.20 \sin 5\theta \\ +\ 6.13 \sin 6\theta \\ -\ 87.51 \sin 7\theta \\ +\ 52.96 \sin 8\theta \\ -\ 16.90 \sin 9\theta \\ -\ 0.02 \sin 10\theta \\ +\ 3.89 \sin 11\theta \\ -\ 1.98 \sin 12\theta \end{array}\right\}.$$

The first of these is simply a transformation of the equation $W = D$ by which e is expressed in terms of θ. From the second, by attributing to ζ in succession the 13 values 0°, 15°, 30°, $\cdots$, 180°, using a tentative process, we can get the corresponding values of θ. Thence by substitution in former results, the corresponding values of the four quantities e, $e \cos \theta$, $e \sin \theta$ and $d\tau/d\theta$ can be obtained. The results follow, the first column containing the argument:

	θ	e	$e\cos\theta$	$e\sin\theta$	$d\tau/d\theta$
0°	0° 0′ 0″.00	0.14746 237	+ 0.14746 237	0.00000 000	18.321384
15	20 27 58.05	0.14664 872	0.13739 211	+ 0.05127 624	18.551258
30	40 25 23.50	0.14418 648	0.10976 569	0.09349 457	19.274084
45	59 23 15.52	0.13999 969	0.07129 163	0.12048 825	20.547277
60	76 57 25.45	0.13402 600	+ 0.03024 713	0.13056 830	22.453723
75	92 51 50.00	0.12634 244	− 0.00631 251	0.12618 464	25.044275
90	107 3 42.77	0.11722 678	0.03439 485	0.11206 745	28.087856
105	119 48 15.87	0.10725 784	0.05331 151	0.09307 055	30.953203
120	131 35 27.90	0.09727 656	0.06457 314	0.07275 327	32.686384
135	143 3 46.20	0.08820 764	0.07050 393	0.05300 740	32.666482
150	154 48 43.11	0.08092 658	0.07323 176	0.03444 156	31.300251
165	167 9 46.91	0.07619 338	0.07428 902	+ 0.01692 847	29.759410
180	180 0 0.00	0.07454 777	− 0.07454 777	0.00000 000	29.112462

From the numbers in the fourth and fifth columns are derived the series:

$$e\cos\theta = \left\{\begin{array}{l} +0.00071\,143 \\ +0.10563\,221\cos\zeta \\ +0.03542\,782\cos 2\zeta \\ +0.00539\,467\cos 3\zeta \\ +0.00031\,839\cos 4\zeta \\ -0.00002\,420\cos 5\zeta \\ -0.00000\,130\cos 6\zeta \\ +0.00000\,245\cos 7\zeta \\ +0.00000\,111\cos 8\zeta \\ +0.00000\,026\cos 9\zeta \\ -0.00000\,044\cos 10\zeta \\ -0.00000\,032\cos 11\zeta \\ +0.00000\,030\cos 12\zeta \end{array}\right\}, \qquad e\sin\theta = \left\{\begin{array}{l} +0.11737\,247\sin\zeta \\ +0.03373\,708\sin 2\zeta \\ +0.00528\,997\sin 3\zeta \\ +0.00035\,675\sin 4\zeta \\ -0.00001\,633\sin 5\zeta \\ -0.00000\,316\sin 6\zeta \\ -0.00000\,095\sin 7\zeta \\ -0.00000\,062\sin 8\zeta \\ +0.00000\,041\sin 9\zeta \\ +0.00000\,018\sin 10\zeta \\ -0.00000\,004\sin 11\zeta \end{array}\right\}.$$

These forms for the integrals of our problem are to be preferred since they can also be used for the case of libration.

XVI.

To complete the solution the periodic series giving the position of the perihelion must be derived. Using logarithms instead of the actual coefficients, the first term of the right member of the third equation of (60) has the expression:

$$\begin{aligned} \frac{2}{\sqrt{(1+\nu)a}}\,a\frac{\partial R}{\partial a} = &\quad [6.9251786] + [7.126034]e^2 - [8.00771]e^4 + [7.9475]e^6 + [8.7115]e^8 \\ &+ \{-[7.9015032] + [8.284709]e^2 + [7.66255]e^4 - [8.9051]e^6\}\, e\cos\theta \\ &+ \{\;\;[8.3084076] - [9.068286]e^2 + [9.55239]e^4 - [9.7559]e^6\}\, e^2\cos 2\theta \\ &+ \{-[8.707357] + [9.66685]\,e^2 - [0.2895]\,e^4\}\, e^3\cos 3\theta \\ &+ \{\;\;[9.104403] - [0.20025]\,e^2 + [0.9097]\,e^4\}\, e^4\cos 4\theta \\ &+ \{-[9.50120] + [0.7008]\,e^2\}\, e^5\cos 5\theta \\ &+ \{\;\;[9.89773] + [1.1811]\,e^2\}\, e^6\cos 6\theta \\ &- \;\;[0.2839]e^7\cos 7\theta \\ &+ \;\;[0.6902]e^8\cos 8\theta\,. \end{aligned}$$

The remaining terms of this expression for $dg/d\tau$ can readily be derived from the values of e and $d\tau/d\theta$ correspondent to the argument ζ which have just been given. Calling these the second part of $dg/d\tau$, we have the following results:

ζ	First Part	Second Part	$dg/d\tau$
0°	+0.00004 9796	−0.00579 1193	−0.00574 1397
15	0.00005 9848	0.00572 9341	0.00566 9493
30	0.00012 1787	0.00554 3667	0.00542 1880
45	0.00023 7477	0.00509 7143	0.00485 9666
60	0.00039 1638	0.00420 4082	0.00381 2444
75	0.00059 5969	0.00270 5347	−0.00210 9378
90	0.00084 3915	−0.00024 2584	+0.00060 1331
105	0.00109 9821	+0.00326 3944	0.00436 3765
120	0.00131 9503	0.00773 7116	0.00905 6619
135	0.00146 5221	0 01284 0064	0.01430 5285
150	0.00153 5401	0.01789 1412	0.01942 6813
165	0.00156 2180	0.02178 6499	0.02334 8679
180	+0.00156 9151	+0.02328 0093	+0.02484 9244

The quantities in the last column furnish the periodic series for $dg/d\tau$. The absolute term shows that the mean motion of the perihelion of the minor planet is 0.00490 0079 times the mean motion of Jupiter. The integration of this series gives the expression for g. These two expressions follow; (g) is the arbitrary constant added to complete the integral, and, in the second term, the unit of t is a Julian year.

$$
\frac{dg}{d\tau}=\left\{\begin{array}{lr}
+0\;00490\;0079 & \\
-0.01441\;6898 & \cos\ \zeta \\
+0\;00444\;6030 & \cos\ 2\zeta \\
-0.00080\;4772 & \cos\ 3\zeta \\
+0.00017\;6760 & \cos\ 4\zeta \\
-0.00006\;7470 & \cos\ 5\zeta \\
+0.00003\;1972 & \cos\ 6\zeta \\
-0.00000\;2614 & \cos\ 7\zeta \\
+0.00000\;0139 & \cos\ 8\zeta \\
-0.00000\;3981 & \cos\ 9\zeta \\
-0.00000\;1706 & \cos 10\zeta \\
+0.00000\;0414 & \cos 11\zeta \\
+0.00000\;0648 & \cos 12\zeta
\end{array}\right\},\qquad
g=\left\{\begin{array}{lr}
(g)+535.''3662\;t & \\
-74645.14 & \sin\ \zeta \\
+11509\;92 & \sin\ 2\zeta \\
-\ \ 1388.93 & \sin\ 3\zeta \\
+\ \ 228.80 & \sin\ 4\zeta \\
-\ \ 69.87 & \sin\ 5\zeta \\
+\ \ 27.59 & \sin\ 6\zeta \\
-\ \ 1.93 & \sin\ 7\zeta \\
+\ \ 0.09 & \sin\ 8\zeta \\
-\ \ 2.27 & \sin\ 9\zeta \\
-\ \ 0.88 & \sin 10\zeta \\
+\ \ 0.19 & \sin 11\zeta \\
+\ \ 0.28 & \sin 12\zeta
\end{array}\right\}.
$$

It is of interest to know the mean motion of the small planet which is not obvious at the beginning of the solution. We have the equation:

$$\frac{d(l+g)}{d\tau}=\frac{d\theta}{d\tau}-\frac{dg}{d\tau}+2\,.$$

Substituting in the right member the mean motions of θ and g, its value is found to be 2.03493 7731; then, if for Jupiter we have $\mu' = 299.''12838$, for the small planet $\mu = 608.''70762$.

XVII.

Illustration in the Case of Libration.—In the example we have chosen to illustrate the theory, libration, when it exists, is always about the value $\theta = 0°$. In addition to the value $\log a = 9.8$ let us assume that the D of (70) is to be so chosen that the half-swing of θ may be 50°. Making, therefore, $\theta = 50°$ in (70), we get the first of the following equations in e, and the second by taking the derivative of the first with respect to e:

$$\left.\begin{array}{ll} -0.00073\ 38426 & e \\ -0.00250\ 58226 & e^2 \\ +0.00439\ 3829 & e^3 \\ +0.58063\ 532 & e^4 \\ -0.02026\ 791 & e^5 \\ +0.17135\ 53 & e^6 \\ -0.01446\ 68 & e^7 \\ -0.32494\ 7 & e^8 \end{array}\right\} = D, \qquad \left.\begin{array}{ll} -0.00073\ 38426 & \\ -0.00501\ 16452 & e \\ +0.01318\ 1487 & e^2 \\ +2.36254\ 128 & e^3 \\ -0.10133\ 955 & e^4 \\ +1.02813\ 18 & e^5 \\ -0.10126\ 76 & e^6 \\ -2.59957\ 6 & e^7 \end{array}\right\} = 0.$$

Both of these equations should be satisfied when θ is at the limit of its swing, viz., when $\theta = \pm 50°$. The root of the second equation which is applicable to our purpose is $e + 0.07606\ 124$, and this value substituted in the first gives $D = -0.00048\ 63102$, which is the value of D which brings about a libration of 50°.

From the equation $\sin\theta = \sin 50° \sin\psi$ we obtain the following corresponding values:

ψ	θ		
°	°	′	″
0	0	0	0.00
15	11	26	8.27
30	22	31	15.64
45	32	47	51.90
60	44	33	38.75
75	47	43	35.34
90	50	0	0.00

By means of these values we determine the form of (70) corresponding to the seven values of ψ. The coefficients are given in the following table (the small figures at the top of the columns denote the order of the final decimal):

ψ	e	e^2	e^3	e^4	e^5	e^6	e^7	e^8
0°	−114 16564^{10}	− 59 0688^{9}	+ 10 210^{8}	+59028 994^{8}	+1214 61^{7}	+ 8372 2^{6}	+ 1414^{5}	+ 8854^{5}
15	111 89923	71 8977	54 217	58936 012	1284 01	8713 9	− 640	15961
30	105 45925	106 9471	164 931	58751 743	1188 52	10353 7	6636	30349
45	95 96627	154 8255	291 587	58665 083	+ 487 32	13434 5	12360	30000
60	85 42473	202 7039	385 516	58769 012	− 671 35	16174 8	11285	+ 4731
75	76 79586	237 7533	429 468	58965 923	1663 22	17134 1	− 4928	−22703
90	− 73 38426	−250 5823	+439 383	+59063 532	−2026 79	+17135 5	+ 1447	−32495

The expressions in this table constitute the left members of 7 equations of the 8th degree; they must be equated to the same quantity $D = -0.00004\ 863102$. The two smallest real roots of each should be derived (they are those suited to our purpose). The connection of these roots with the variable ψ is settled in following way: the larger of the two roots is made to correspond to the value of ψ standing as the argument in the table, while the smaller is assigned to the value $180° - \psi$; the two roots being equal for $\psi = 90°$, the common value is assigned to that value of ψ. This arrangement is made in order that ψ and τ may augment together. These values of e together with the corresponding values of θ are, in succession, substituted in (65); thus we have the values of $d\tau/d\psi$ corresponding to equidistant values of ψ. These resu[illegible] contained in the following table:

ψ	e	$d\tau/d\psi$
0°	0.10846 187	37.23986
15	0.10765 795	37.70988
30	0.10518 818	39.14539
45	0.10088 191	41.57020
60	0.09452 189	44.76280
75	0.08605 349	47.65995
90	0.07606 124	48.00164
105	0.06601 592	44.40759
120	0.05744 757	38.68685
135	0.05102 653	33.34281
150	0.04671 683	29.47279
165	0.04426 767	27.22320
180	0.04347 566	26.49213

It should be noted that, in the computation of the third column, the factor $\cos\psi/(\partial W/\partial e)$ takes on the indeterminate form 0/0; employing the usual method of treating vanishing fractions, this factor equals $-1/[(\partial^2 W/\partial e^2)(de/d\psi)]$. If here we should use the equation $W = D$ to determine $de/d\psi$, the result would again be indeterminate. But this difficulty is avoided by employing the value of e as a periodic function of ψ given by the quantities of the second column. Thus if

$$e = a_0 + a_1 \cos\psi + a_2 \cos 2\psi + a_3 \cos 3\psi + \cdots,$$

then

$$\frac{de}{d\psi} = - a_1 \sin\psi - 2a_2 \sin 2\psi - 3a_3 \sin 3\psi - \cdots,$$

and, for the special value $\psi = 90°$, this becomes:

$$\frac{de}{d\psi} = - a_1 + 3a_3 - 5a_5 + 7a_7 - \cdots.$$

For this special value of ψ it is found that

$$e\frac{\partial^2 W}{\partial e^2} = + 0.00288\ 92836, \qquad \frac{de}{d\psi} = - 0.03939\ 373.$$

The mean of the numbers in the last column of the table, allowing half weight to the first and last, is 38.65409, which is the number of revolutions of Jupiter contained in the period of libration; thus this period is 458.5144 Julian years. From the special values of e and $d\tau/d\psi$ given in the table we derive the periodic series representing them. The latter can be integrated, and, as before, we put ζ for $\theta_0(t + c)$. Thus we get the following expressions:

$$e=\left\{\begin{array}{l} 0.0759\ 8399 \\ +0.0338\ 8957 \cos\ \psi \\ -0.0000\ 4144 \cos\ 2\psi \\ -0.0015\ 2988 \cos\ 3\psi \\ +0.0000\ 3031 \cos\ 4\psi \\ +0.0001\ 5016 \cos\ 5\psi \\ -0.0000\ 0467 \cos\ 6\psi \\ -0.0000\ 1929 \cos\ 7\psi \\ +0.0000\ 0062 \cos\ 8\psi \\ +0.0000\ 0281 \cos\ 9\psi \\ -0.0000\ 0012 \cos 10\psi \\ -0.0000\ 0026 \cos 11\psi \\ +0.0000\ 0008 \cos 12\psi \end{array}\right\}, \qquad \zeta=\psi+\left\{\begin{array}{l} +29862''.37 \sin\ \psi \\ -20922.46 \sin\ 2\psi \\ -\ \ 416.31 \sin\ 3\psi \\ +\ 1650.00 \sin\ 4\psi \\ +\ \ \ 13.01 \sin\ 5\psi \\ -\ \ 193.31 \sin\ 6\psi \\ -\ \ \ \ 0.37 \sin\ 7\psi \\ +\ \ \ 27.39 \sin\ 8\psi \\ +\ \ \ \ 0.01 \sin\ 9\psi \\ -\ \ \ \ 4.65 \sin 10\psi \\ -\ \ \ \ 0.02 \sin 11\psi \\ +\ \ \ \ 0.81 \sin 12\psi \end{array}\right\}.$$

The first of these is simply a transformation of the equation $W = D$, by which e is expressed in terms of the auxiliary variable ψ.

Attributing to ζ, in succession, the 13 values $0°$, $15°$, $30°$, $\cdots$, $180°$, by a tentative process we can get the corresponding values of ψ, as also, by substitution those of e, $e \cos\theta$ and $e \sin\theta$. These results follow:

ζ	ψ	e	$e \cos\theta$	$e \sin\theta$
0°	0° 0′ 0″.00	0.1084 6187	+ 0.1084 6187	0.0000 0000
15	15 30 0.64	0.1068 0283	0.1045 4084	+ 0.0218 6455
30	30 35 58.93	0.1022 4969	0.0941 5541	0.0398 7176
45	44 58 46.75	0.0956 9588	0.0804 5265	0.0518 1767
60	58 28 33.58	0.0879 7453	0.0666 2875	0.0574 4671
75	71 8 20.78	0.0797 1930	0.0549 1390	0.0577 8952
90	83 15 23.39	0.0714 4655	0.0463 7246	0.0543 5260
105	95 20 46.24	0.0636 2489	0.0411 4864	0.0485 2749
120	108 7 15.59	0.0566 9409	0.0388 6515	0.0412 7614
135	122 25 27.98	0.0510 7994	0.0389 6451	0.0330 2919
150	139 5 9.66	0.0468 5171	0.0405 2863	0.0235 0559
165	158 31 3.57	0.0443 2198	0.0425 4216	+ 0.0124 3393
180	180 0 0.00	0.0434 7566	+ 0.0434 7566	0.0000 0000

From the data of the fourth and fifth columns result the periodic series:

$$e\cos\theta=\left\{\begin{array}{l} +0.0604\ 2349 \\ +0.0309\ 3374 \cos\ \zeta \\ +0.0147\ 2940 \cos\ 2\zeta \\ +0.0015\ 8319 \cos\ 3\zeta \\ +0.0007\ 3464 \cos\ 4\zeta \\ -0.0000\ 1013 \cos\ 5\zeta \\ +0.0000\ 6769 \cos\ 6\zeta \\ -0.0000\ 1229 \cos\ 7\zeta \\ +0.0000\ 1611 \cos\ 8\zeta \\ -0.0000\ 0669 \cos\ 9\zeta \\ +0.0000\ 0106 \cos 10\zeta \\ +0.0000\ 0528 \cos 11\zeta \\ -0.0000\ 0363 \cos 12\zeta \end{array}\right\}, \qquad e\sin\theta=\left\{\begin{array}{l} +0.0571\ 8416 \sin\ \zeta \\ +0.0093\ 8541 \sin\ 2\zeta \\ +0.0030\ 1596 \sin\ 3\zeta \\ +0.0000\ 5256 \sin\ 4\zeta \\ +0.0001\ 9928 \sin\ 5\zeta \\ -0.0000\ 1597 \sin\ 6\zeta \\ +0.0000\ 1768 \sin\ 7\zeta \\ -0.0000\ 0390 \sin\ 8\zeta \\ +0.0000\ 0771 \sin\ 9\zeta \\ -0.0000\ 0714 \sin 10\zeta \\ +0.0000\ 0492 \sin 11\zeta \end{array}\right\}.$$

XVIII.

In computing, for this case, the values of $dg/d\tau$ by the third equation of (60) we make a like division into two parts as in the former case. Substituting the values of e and θ which correspond to the values 0°, 15°, 30°, $\cdots$, 180° of ζ we get the following special values:

ζ	First Part	Second Part	$dg/d\tau$
0°	+ 0.00019 583	— 0.05008 981	— 0.04989 398
15	20 725	4843 252	4822 527
30	24 656	4385 892	4361 236
45	30 013	3722 995	3692 982
60	36 434	2938 834	2902 400
75	42 685	2098 623	2055 938
90	47 979	1247 567	1199 588
105	51 817	— 0.00410 869	— 0.00359 052
120	54 013	+ 0.00397 237	+ 0.00451 250
135	54 781	1211 642	1266 423
150	54 497	1991 948	2046 445
165	53 746	2629 903	2683 649
180	+ 0.00053 360	+ 0.02891 583	+ 0.02944 943

From the quantities in the last column we obtain the periodic series for $dg/d\tau$, and thence by integration the expression for g; these results follow:

$$\frac{dg}{d\tau}=\left\{\begin{array}{lr} -0.0116\ 3174 & \\ -0.0373\ 2443 & \cos\ \zeta \\ +0.0007\ 9929 & \cos\ 2\zeta \\ -0.0020\ 0735 & \cos\ 3\zeta \\ +0.0005\ 0972 & \cos\ 4\zeta \\ -0.0002\ 9143 & \cos\ 5\zeta \\ +0.0000\ 8500 & \cos\ 6\zeta \\ -0.0000\ 4117 & \cos\ 7\zeta \\ +0.0000\ 1080 & \cos\ 8\zeta \\ -0.0000\ 0438 & \cos\ 9\zeta \\ +0.0000\ 0251 & \cos 10\zeta \\ -0.0000\ 0293 & \cos 11\zeta \\ +0.0000\ 0214 & \cos 12\zeta \end{array}\right\}, \qquad g=\left\{\begin{array}{lrr} (g) - 1270''.844\ t & & \\ -\ 297586.8 & \sin & \zeta \\ +\ \ 3186.4 & \sin & 2\zeta \\ -\ \ 5334.9 & \sin & 3\zeta \\ +\ \ 1016.0 & \sin & 4\zeta \\ -\ \ 464.7 & \sin & 5\zeta \\ +\ \ 113.0 & \sin & 6\zeta \\ -\ \ 46.9 & \sin & 7\zeta \\ +\ \ 10.8 & \sin & 8\zeta \\ -\ \ 3.9 & \sin & 9\zeta \\ +\ \ 2.0 & \sin & 10\zeta \\ -\ \ 2.1 & \sin & 11\zeta \\ +\ \ 1.4 & \sin & 12\zeta \end{array}\right\}.$$

The unit of t in the second expression is a Julian year.

By using the same formula as in the former case we find that the mean μ of the small planet in this case has the value $609''.47474$.

XIX.

In attempting to apply the preceding method to the case where $D=0$, we should find that $dg/d\tau$ became infinite at the point where $\theta=0^\circ$ or $\theta=180^\circ$, and, when D is quite small, we should have to deal with inconveniently large numbers. This difficulty is surmounted by computing the differentials of the two quantities $e\cos g$ and $e\sin g$ in place of that of g.